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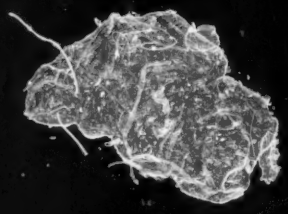

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
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A Sufficient Condition for Matrix Stability*

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(April 22, 1974)

An n by n complex matrix A is said to be positive stable if Re $(\lambda) > 0$ for each eigenvalue λ of A. If A satisfies both of the following two conditions, then A is positive stable: (1) for each $k=1, \ldots, n$, the real part of the sum of the k by k principal minors of A is positive; and (2) the minimum of the real parts of the eigenvalues of A is itself an eigenvalue of A. Special cases include hermitian positive definite matrices and M-matrices.

Key words: M-matrix; stable matrix; principal minors.

Suppose $A=(a_{ij}) \epsilon M_n(C)$, the n by n complex matrices. Then let $P_k(A)$ denote the sum of the k by k principal minors of A. If Re $(\lambda) > 0$ for each eigenvalue λ of A, then A is called (positive) *stable*, and if Re $[P_k(A)] > 0$, $k=1, \ldots, n$, then A is called *prestable* [2].[1] This note was suggested by the following question which arose in the stability analysis of an economic equilibrium [3]. Suppose $A \epsilon M_n(R)$, the real n by n matrices, has the sign pattern (M):

$$a_{ii} > 0, \qquad i=1, \ldots, n$$

and

$$a_{ij} \leq 0, \qquad i \neq j,\ i, j=1, \ldots, n.$$

If A *is prestable, does it then follow that* A *is stable?* Unfortunately, this is not one of the 13 equivalent conditions given in [1] for A to be an M-matrix. An affirmative answer to this question is a corollary to what we shall prove.

LEMMA 1: *If* $A \epsilon M_n(C)$ *is prestable, then* A *has no nonpositive real eigenvalues.*

PROOF: Suppose A is prestable and that $-r$, $r \geq 0$, is a nonpositive real eigenvalue of A. But then $A-(-r)I=A+rI$ is singular which means det $(A+rI)=0$. However,

$$\det (A+rI) = r^n + \sum_{k=1}^{n} P_k(A) r^{(n-k)}$$

so that Re$[\det(A+rI)] > 0$, a contradiction. This means there is no such r and completes the proof.

When the minimum of the real parts of the eigenvalues of $A \epsilon M_n(C)$ is itself an eigenvalue of A, we shall say that A has the *property* (*).

THEOREM: *Suppose* $A \epsilon M_n(C)$ *has property* (*). *If* A *is prestable, then* A *is stable.*

PROOF: Because of lemma 1 and property (*), the minimum of the real parts of the eigenvalues of A is positive. This means A is stable.

Since it is clear that the well known class of hermitian matrices satisfies property (*), it follows that:

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[1] Figures in brackets indicate the literature references at the end of this paper.

COROLLARY 1: *If* $A \epsilon M_n(C)$ *is hermitian, then* A *prestable implies* A *stable.*

Another well known class of matrices which has property (*) are those of sign pattern (M).

LEMMA 2: *If* $A \epsilon M_n(R)$ *has sign pattern* (M), *then* A *has property* (*).

PROOF: Suppose that m is a real number greater than each of the a_{ii}. Then A may be written

$$A = mI - P$$

where P is an entry-wise nonnegative matrix. Because of the Perron-Frobenius theorem, P has a dominant positive eigenvalue r such that $r \geq |\lambda|$, and thus $r \geq \text{Re}(\lambda)$, for each eigenvalue λ of P. It then follows that $m - r$ is an eigenvalue of A and is the smallest of the real parts of the eigenvalues of A, so that A has property (*).

COROLLARY 2: *If* $A \epsilon M_n(R)$ *has sign pattern* (M), *then* A *prestable implies* A *stable.*

PROOF: This follows from Lemma 2 and the theorem.

We close with a question of interest which arises from the preceding remarks:

What are necessary and sufficient conditions on A *such that* A *satisfy property* (*)*?*

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(Paper 78B3–405)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
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Computation of the Field of Values of a 2 X 2 Matrix*

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(May 17, 1974)

It is known that the field of values, $F(A) = \{x^*Ax : x^*x = 1,\ x \epsilon C^2\}$, of a 2×2 matrix A is a convex set whose boundary is an ellipse. This is used to compute $F(A)$ explicitly from the entries of A when A is 2×2 and real. Employing known properties of the field of values, this may then be used to estimate $F(A)$ when A is $n \times n$ and real.

Key words: Eigenvalues; ellipse; field of values; square matrix.

For an arbitrary $n \times n$ complex matrix A, the *field of values*, $F(A)$ is defined by

$$F(A) = \{x^*Ax : x^*x = 1, \quad x \epsilon C^n\}.$$

It is well known that $F(A)$ is a closed, convex subset of the complex plane. It arises in several practical and theoretical settings, and, in particular, $F(A)$ contains each eigenvalue of A. Furthermore, in case $n = 2$, the boundary of $F(A)$ is an ellipse (or, in degenerate cases, a line segment or circle). In [1][1] a method is given for determining this ellipse by first solving for the eigenvectors of A. In this note we determine the ellipse directly as a function of the entries for any real (2×2) matrix A.

THEOREM: *Suppose* $\mathrm{A} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, a, b, c, d *real, and let* x, y, z, w *be the four points in the complex plane:*

$$\mathrm{x} = \frac{1}{2}\left(\mathrm{a} + \mathrm{d} - ((\mathrm{a} - \mathrm{d})^2 + (\mathrm{b} + \mathrm{c})^2)^{1/2}\right)$$

$$\mathrm{y} = \frac{1}{2}\left(\mathrm{a} + \mathrm{d} + ((\mathrm{a} - \mathrm{d})^2 + (\mathrm{b} + \mathrm{c})^2)^{1/2}\right)$$

$$\mathrm{w} = \frac{\mathrm{a} + \mathrm{d}}{2} + \mathrm{i}\left|\frac{\mathrm{b} - \mathrm{c}}{2}\right|$$

$$\mathrm{z} = \frac{\mathrm{a} + \mathrm{d}}{2} - \mathrm{i}\left|\frac{\mathrm{b} - \mathrm{c}}{2}\right|.$$

Then the boundary of the convex set F(A) *is the (possibly degenerate) ellipse whose axes are the line segments joining* x *to* y *and* w *to* z *respectively.*

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*This work was done while the author was a National Academy of Sciences–National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

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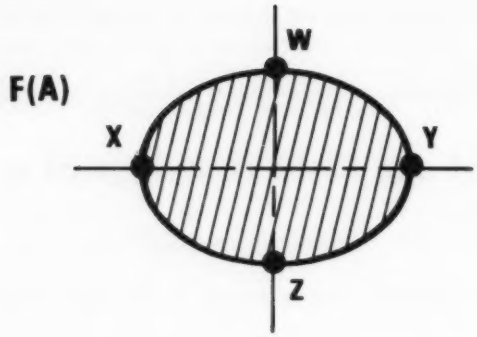
[1] Figures in brackets indicate the literature references at the end of this paper.



PROOF: We know that the boundary of $F(A)$ is an ellipse. Since A is real, the ellipse is symmetric with respect to the real axis of the complex plane. Therefore, one of the axes of the ellipse lies on the real line. It is the line segment whose endpoints are the eigenvalues of

$$\text{Re } (A)=\frac{A+A^T}{2}=\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}.$$

A computation yields that these two values are the numbers x and y. The remaining axis of the ellipse must be parallel to the imaginary axis of the complex plane. The length of this axis is just the distance between the eigenvalues of

$$\text{Im } (A)=\frac{A-A^T}{2}=\begin{bmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{bmatrix}.$$

This implies that the second axis has endpoints w and z as asserted. This completely determines the boundary ellipse of $F(A)$ and thus $F(A)$ itself.

In case (1) $b=c$ (A symmetric) or (2) $a=d=0$ and $c=-b$ (A skew-symmetric), the given ellipse degenerates into a line segment. In case $(a-d)^2+4bc=0$, the ellipse degenerates into a circle and A has only a single repeated eigenvalue $\lambda=\frac{a+d}{2}$. In all other cases the boundary of $F(A)$ is a legitimate ellipse.

For an arbitrary $n\times n$ matrix A, the normalization

$$B=A-\frac{\text{Tr } (A)}{n}I$$

produces a matrix B of trace 0 whose eigenvalues and field of values are merely a translation of those of A by the scalar $\frac{\text{Tr } (A)}{n}$. For a real 2×2 matrix A, this normalization allows us to give the equation of the boundary ellipse of $F(A)$ in standard form, since the translated ellipse will be centered at the origin.

COROLLARY 1: *Suppose* A *is* 2×2, *real*, $\mathrm{Tr}(A)=0$, *and* $A=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

Then the boundary of $F(A)$ *has one of the following forms:*

(*i*) *The line segment joining*

$$\pm [a^2+b^2]^{1/2} \text{ if } c=b;$$

(*ii*) *The line segment joining*

$$\pm i\left|\frac{b-c}{2}\right| \text{ if } a=0 \text{ and } c=-b; \text{ and}$$

(*iii*) $\left\{u+iv : \frac{u^2}{u_0^2}+\frac{v^2}{v_0^2}=1\right\}$ *where*

$$u_0=\left[a^2+\left(\frac{b+c}{2}\right)^2\right]^{1/2},\ v_0=\left|\frac{b-c}{2}\right|;\ u, v \text{ real},$$

otherwise.

A computation then shows that in case (i) and (ii), the eigenvalues of A occur at the endpoints of the given line segments. In case (iii) the eigenvalues of A occur at the foci (center) of the given ellipse (circle). In each of the cases, the eigenvalues of A are

$$\pm (a^2+bc)^{1/2}.$$

For arbitrary n, the theorem above and the convexity of $F(A)$ may be used to obtain an "inner" approximation of $F(A)$ for an arbitrary real $n\times n$ matrix $A=(a_{ij})$. Let

$$A_{i,j}=\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$$

denote the 2×2 principal submatrix of A determined by i and j. Then $F(A_{i,j})$ is determined explicitly by the theorem when A is real. Let also Co(S) denote the closed convex hull of a subset S of the complex plane. We then have:

COROLLARY 2: *Suppose* $A=(a_{ij})$ *is* $n\times n$, *then*

$$\mathrm{Co}\left(\bigcup_{1\le i<j\le n} F(A_{i,j})\right)\subseteq F(A).$$

PROOF: The assertion is valid since the field of values of any principal submatrix of A is contained in $F(A)$ and $F(A)$ is convex. Its efficacy lies in the fact that the left hand side is easily computed from the entries of A when A is real.

An outer approximation for $F(A)$ was developed in [2, 3] using Gersgorin estimates. Thus by combining our corollary 2 and the main theorem of [3], convex sets $s(A)$ and $S(A)$ may be computed directly from the entries of A so that

$$s(A)\subseteq F(A)\subseteq S(A).$$

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(Paper 78B3–406)

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The Factorization of a Matrix as the Commutator of Two Matrices

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(April 4, 1974)

Let $P=I_p \dot{+} (-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. The following theorem is proved.

THEOREM: *If* $X=cZ$ *where* Z *is a* 4×4 P*-orthogonal,* P*-skew-symmetric matrix and* $|c|\leq 2$, *there exist matrices* A *and* B, *both of which are* P*-orthogonal and* P*-skew-symmetric, such that* $X=AB-BA$. Methods for obtaining certain matrices which satisfy $X=AB-BA$ are given. Methods are also given for determining pairs of anticommuting P-orthogonal, P-skew-symmetric matrices.

Key words: Anticommuting; commutator; factorization; matrix; orthogonal; skew-symmetric.

1. Introduction

Let $P=I_p \dot{+} (-I_q)$, the direct sum of the $p\times p$ identity matrix and the negative of the $q\times q$ identity matrix. Katz and Olkin [2] [1] define a real matrix A to be orthogonal with respect to P (P-orthogonal) if and only if

$$APA'=P \tag{1}$$

where A' is the transpose of A. Furthermore, they define B to be skew-symmetric with respect to P (P-skew-symmetric) if and only if BP is skew-symmetric in the ordinary sense.

The main result of this paper is concerned with matrices which are both P-orthogonal and P-skew-symmetric of order $n=4=p+q$. Smith [7] proved that such matrices exist in only two cases, $p=4$, $q=0$ and $p=q=2$. In the first case P-orthogonal and P-skew-symmetric reduce to orthogonal and skew-symmetric in the ordinary sense.

Pearl [4] and Smith [6] proved the following theorem in the cases $p=4$, $q=0$ and $p=q=2$ respectively.

THEOREM 1: *If the* 4×4 *matrices* A *and* B *are both* P*-orthogonal and* P*-skew-symmetric then their commutator,* $[A, B]\equiv AB-BA$, *is a scaler multiple of a* 4×4 P*-orthogonal,* P*-skew-symmetric matrix.*

The purpose of this paper is to prove a converse to Theorem 1. Shoda [5] proved that if X is a square matrix with zero trace having elements in an algebraically closed field then there exist matrices A and B such that $X=AB-BA$. Albert and Muckenhoupt [1] removed the restriction that the field be algebraically closed. However, both the method of Shoda and the method of Albert and Muckenhoupt give a singular matrix B. The main result of this paper is:

THEOREM 2: *If* $X=cZ$ *where* Z *is a* 4×4 P*-orthogonal,* P*-skew-symmetric matrix and* $|c|\leq 2$, *there exist matrices* A *and* B, *both of which are* P*-orthogonal and* P*-skew-symmetric, such that* $X=AB-BA$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Methods for obtaining certain matrices which satisfy $X=AB-BA$ are given. Methods are also given for determining pairs of anticommuting P-orthogonal, P-skew-symmetric matrices.

2. Anticommuting Matrices

In examining the structure of P-orthogonal, P-skew-symmetric matrices in the case $p=4$, $q=0$, Pearl [4] shows that any such matrix has exactly one of the following forms:

$$\text{(i)}\ \alpha_1 R_1+\alpha_2 R_2+\alpha_3 R_3, \qquad \alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

$$\text{(ii)}\ \alpha_1 S_1+\alpha_2 S_2+\alpha_3 S_3, \qquad \alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

where the α_i are real scalers and the R_i and S_i are the first and second regular representations respectively of the real quaternions [3].

Similarly, in the case $p=q=2$, Smith [6] shows that any such matrix has exactly one of the following forms:

$$\text{(iii)}\ \alpha_1 R_1 P+\alpha_2 S_2 P+\alpha_3 S_3 P, \qquad \alpha_2^2+\alpha_3^2-\alpha_1^2=1$$

$$\text{(iv)}\ \alpha_1 S_1 P+\alpha_2 R_2 P+\alpha_3 R_3 P, \qquad \alpha_1^2-\alpha_2^2-\alpha_3^2=-1$$

where $P=I_2 \dot{+} (-I_2)$.

A further examination of these papers leads to

THEOREM 3: *If* Z *is a* 4×4 P*-orthogonal,* P*-skew-symmetric matrix there exists a* 4×4 P*-orthogonal,* P*-skew-symmetric matrix* B *such that* ZB $=-$ BZ.

PROOF: There are four cases to consider.

Case 1, $Z=\alpha_1 R_1+\alpha_2 R_2+\alpha_3 R_3$. If $\alpha_3\neq 0$, choose arbitrary β_1', β_2' and set

$$\beta_3'=-\frac{1}{\alpha_3}(\alpha_1\beta_1'+\alpha_2\beta_2').$$

Let $x=\beta_1'^2+\beta_2'^2+\beta_3'^2$ and set $\beta_i=\dfrac{\beta_i'}{\sqrt{x}}$, $i=1,2,3$. If $\alpha_3=0$, let $\beta_1=\beta_2=0$ and $\beta_3=1$. Clearly, in either situation

$$\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3=0 \tag{2}$$

and

$$\beta_1^2+\beta_2^2+\beta_3^2=1. \tag{3}$$

Letting $B=\beta_1 R_1+\beta_2 R_2+\beta_3 R_3$, by (3) B is P-orthogonal, P-skew-symmetric and by (2)

$$ZB=-BZ.$$

Case 2, $Z=\alpha_1 S_1+\alpha_2 S_2+\alpha_3 S_3$. Choose β_i, $i=1,2,3$ as in Case 1 and let

$$B=\beta_1 S_1+\beta_2 S_2+\beta_3 S_3.$$

Case 3, $Z=\alpha_1 R_1 P+\alpha_2 S_2 P+\alpha_3 S_3 P$. The matrix $B=\beta_1 R_1 P+\beta_2 S_2 P+\beta_3 S_3 P$ will be P-orthogonal, P-skew-symmetric if

$$\beta_2^2+\beta_3^2-\beta_1^2=1 \tag{4}$$

and $ZB=-BZ$ if

$$\alpha_2 \beta_2 + \alpha_3 \beta_3 - \alpha_1 \beta_1 = 0. \tag{5}$$

If $\alpha_1 + \alpha_2 \neq 0$, set $\beta_1 = \dfrac{\alpha_3}{\alpha_1 + \alpha_2}$, $\beta_2 = \dfrac{-\alpha_3}{\alpha_1 + \alpha_2}$, and $\beta_3 = 1$. Clearly, (4) and (5) are satisfied. If $\alpha_1 = \alpha_2 = 0$, set $\beta_1 = \beta_3 = 0$, $\beta_2 = 1$. Again (4) and (5) are satisfied. If $\alpha_1 = -\alpha_2 \neq 0$, since clearly $\alpha_3 = \pm 1$, set $x = \dfrac{1}{1 + \alpha_1^2}$ and let $\beta_1 = 0$, $\beta_3 = \alpha_3 \alpha_1 \sqrt{x}$, $\beta_2 = \sqrt{x}$. Again (4) and (5) are satisfied.

Case 4, $Z = \alpha_1 S_1 P + \alpha_2 R_2 P + \alpha_3 R_3 P$. Let $B = \beta_1 S_1 P + \beta_2 R_2 P + \beta_3 R_3 P$ where the β_i are chosen as in case 3.

3. Proof of Theorem 2

In order to prove Theorem 2 it is convenient to first prove the following lemmas.

LEMMA 1: (i) *If* B *is* P*-skew-symmetric then* $B' = -PBP$.

(ii) *If* B *is* P*-skew-symmetric and* P*-orthogonal than* $B^2 = -I$.

LEMMA 2: *If* Z *is a* P*-orthogonal,* P*-skew-symmetric matrix and* $|c| \leq 2$ *then*

$$Y = \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z \text{ is P-orthogonal.}$$

PROOF: By direct computation,

$$YPY' = \left(\frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z \right) P \left(\frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z' \right)$$

$$= \frac{4 - c^2}{4} P + \frac{c^2}{4} ZPZ' + \frac{\sqrt{4 - c^2}}{2} (ZP + PZ').$$

However, $ZPZ' = P$ by (1) and by Lemma 1

$$ZP + PZ' = ZP + P(-PZP) = ZP - ZP = 0.$$

Thus $YPY' = \dfrac{4 - c^2}{4} P + \dfrac{c^2}{4} P + O = P$ and by (1) Y is P-orthogonal.

LEMMA 3: *If* Z *is* P*-orthogonal,* P*-skew-symmetric and* $|c| \leq 2$, *and if* B *is* P*-orthogonal,* P*-skew-symmetric such that* $ZB = -BZ$, *then* $A = \left(\dfrac{\sqrt{4 - c^2}}{2} I + \dfrac{c}{2} Z \right) PB'P$ *satisfies* $[A,B] = cZ$.

PROOF: $AB = (YPB'P)\, B = (YP)\, (B'PB) = (YP)\, P = Y$

$$= \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} Z$$

$$BA = B\,(YPB'P) = \frac{\sqrt{4 - c^2}}{2} BPB'P + \frac{c}{2} BZPB'P$$

$$= \frac{\sqrt{4 - c^2}}{2} I + \frac{c}{2} BZPB'P$$

$$= \frac{\sqrt{4 - c^2}}{2} I - \frac{c}{2} ZBPB'P$$

$$= \frac{\sqrt{4-c^2}}{2} I - \frac{c}{2} Z.$$

Thus $[A,B] = AB - BA = cZ$.

COROLLARY: *The matrix* A *defined in Lemma 3 is* P*-orthogonal,* P*-skew-symmetric.*

PROOF: By Lemmas 1 and 2 A is the product of two P-orthogonal matrices. Hence A is P-orthogonal. Also

$$A = \left(\frac{\sqrt{4-c^2}}{2} I + \frac{c}{2} Z \right) PB'P$$

$$= \frac{\sqrt{4-c^2}}{2} PB'P + \frac{c}{2} ZPB'P.$$

By Lemma 1, $\frac{\sqrt{4-c^2}}{2} PB'P = -\frac{\sqrt{4-c^2}}{2} B$ which is P-skew-symmetric. Furthermore

$$\left(\frac{c}{2} ZPB'P\right)' = \frac{c}{2} PBPZ'$$

$$= -\frac{c}{2} B'Z' \qquad \text{by Lemma 1}$$

$$= \frac{c}{2} Z'B' \qquad \text{since } ZB = -BZ$$

$$= -\frac{c}{2} PZPB' \qquad \text{by Lemma 1}$$

$$= -\frac{c}{2} P\left(ZPB'P\right)P.$$

Thus A is the sum of two P-skew-symmetric matrices and hence A is P-skew-symmetric.

In the 4×4 case, the existence of the matrix B is given by Theorem 3. Thus Theorem 3, Lemma 3, and the Corollary complete the proof of Theorem 2.

4. Conclusion

Theorem 2 provides a converse to the theorems of Pearl [4] and Smith [6]. While Theorem 2 is restricted to the 4×4 case, the results of section 3 refer to the general $n \times n$ case. Smith [8] has generalized Theorem 1 to the $n \times n$ case. Perhaps the results of section 3 can be applied to find a converse of that result.

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Complete Elliptic Integrals Resulting from Infinite Integrals of Bessel Functions*

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Infinite integrals of Bessel and modified Bessel functions reducible to complete elliptic integrals are compiled. These formulas are of great use in solving problems of applied mathematics, physics and engineering.

Key words: Applied mathematics; Bessel functions; complete elliptic integrals; engineering; infinite integrals; modified Bessel functions; physics; signal statistics.

AMS Subject Classification: 33A25, 33A40, 44A20.

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1. Introduction

This table, which is an outgrowth of the author's experience in the theoretical study of signal statistics, contains infinite integrals of Bessel and modified Bessel functions reducible to complete elliptic integrals.[1] The formulas listed below are those important in applications and all results are expressed in conviently compact forms.

The materials were first extracted from the author's own memorandum which were then thoroughly augmented and rearranged in the present form by scrutinizing various published books and papers.

The parameters used in this table are usually positive real and notations occurring several times on a section are explained at the top of the section.

2. Integrands Involving Bessel Functions of the First Kind

2.1. $k^2=\dfrac{\sqrt{p^2+a^2}-p}{2\sqrt{p^2+a^2}}$

$$\int_0^\infty e^{-px^2}J_0(ax^2)dx=\frac{\sqrt{1-2k^2}}{\sqrt{\pi p}}K(k). \qquad \text{OL 119 (13.2)} \qquad (1)$$

$$\int_0^\infty e^{-px^2}J_0(ax^2)x^2dx=\frac{\sqrt{(1-2k^2)^3}}{2\sqrt{\pi p^3}}\,[2E(k)-K(k)]. \qquad \text{OL 119 (13.1)} \qquad (2)$$

$$\int_0^\infty e^{-px^2}J_0(ax^2)x^4dx=\frac{\sqrt{(1-2k^2)^5}}{4\sqrt{\pi p^5}}\,[8(1-2k^2)E(k)-(5-8k^2)K(k)]. \qquad (3)$$

[1] Including Bierens de Haan, D., Nouvelles Tables d'Intégrales Définies (Leide, Amsterdam 1867), there exists a considerable literature on complete elliptic integrals involving algebraic, trigonometric, hyperbolic or logarithmic integrands.

$$\int_0^\infty e^{-px^2}J_0(ax^2)x^6dx=\frac{\sqrt{(1-2k^2)^7}}{8\sqrt{\pi p^7}}[2(23-128k^2+128k^4)E(k)-(31-144k^2+128k^4)K(k)]. \tag{4}$$

$$\int_0^\infty e^{-px^2}J_1(ax^2)dx=\frac{\sqrt{1-2k^2}}{k\sqrt{\pi p(1-k^2)}}[E(k)-(1-k^2)K(k)]. \tag{5}$$

$$\int_0^\infty e^{-px^2}J_1(ax^2)x^2dx=\frac{\sqrt{(1-2k^2)^3}}{2k\sqrt{\pi p^3(1-k^2)}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{6}$$

$$\int_0^\infty e^{-px^2}J_1(ax^2)x^4dx=\frac{\sqrt{(1-2k^2)^5}}{4k\sqrt{\pi p^5(1-k^2)}}[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)]. \tag{7}$$

$$\int_0^\infty e^{-px^2}J_1(ax^2)x^{-2}dx=\frac{2\sqrt{p}}{3k\sqrt{\pi(1-k^2)(1-2k^2)}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{8}$$

$$\int_0^\infty e^{-px^2}J_2(ax^2)dx=\frac{\sqrt{1-2k^2}}{3k^2(1-k^2)\sqrt{\pi p}}[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \tag{9}$$

$$\int_0^\infty e^{-px^2}J_2(ax^2)x^2dx=\frac{\sqrt{(1-2k^2)^3}}{2k^2(1-k^2)\sqrt{\pi p^3}}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{10}$$

$$\int_0^\infty e^{-px^2}J_2(ax^2)x^4dx=\frac{\sqrt{(1-2k^2)^5}}{4k^2(1-k^2)\sqrt{\pi p^5}}[(2+5k^2-8k^4)\times(1-k^2)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \tag{11}$$

$$\int_0^\infty e^{-px^2}J_2(ax^2)x^6dx=\frac{\sqrt{(1-2k^2)^7}}{8k^2(1-k^2)\sqrt{\pi p^7}}[(2+15k^2-144k^4+128k^6)\times(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]. \tag{12}$$

$$\int_0^\infty e^{-px^2}J_2(ax^2)x^{-2}dx=\frac{2\sqrt{p}}{15k^2(1-k^2)\sqrt{\pi(1-2k^2)}}\times[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{13}$$

$$\int_0^\infty e^{-px^2}J_3(ax^2)dx=\frac{\sqrt{1-2k^2}}{15k^3\sqrt{\pi p(1-k^2)^3}}[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]. \tag{14}$$

$$\int_0^\infty e^{-px^2}J_3(ax^2)x^2dx=\frac{\sqrt{(1-2k^2)^3}}{6k^3\sqrt{\pi p^3(1-k^2)^3}}[(8-15k^2+3k^4)\times(1-k^2)K(k)-(8-19k^2+9k^4-6k^6)E(k)]. \tag{15}$$

$$\int_0^\infty e^{-px^2} J_3(ax^2) x^{-2} dx = \frac{2\sqrt{p}}{105k^3\sqrt{\pi(1-k^2)^3(1-2k^2)}} [\,(1-k^2)$$

$$\times (8-15k^2+3k^4)K(k) - (8-19k^2+9k^4-6k^6)E(k)\,]. \qquad (16)$$

2.2. $k_1^2 = \frac{1}{2}(1-\sqrt{1-a^2\gamma^2})$, $\quad k_2^2 = \frac{1}{2}(1-\sqrt{1-b^2\gamma^2})$,

$$\gamma^2 = \frac{2}{p^2+a^2+b^2+\sqrt{(p^2+a^2+b^2)^2-4a^2b^2}}$$

$$\int_0^\infty e^{-px^2} J_0(ax^2) J_0(bx^2) dx = \frac{2\sqrt{\gamma}}{\sqrt{\pi^3}} K(k_1) K(k_2). \qquad (1)$$

$$\int_0^\infty e^{-px^2} J_1(ax^2) J_0(bx^2) dx = \frac{2\sqrt{\gamma}}{k_1\sqrt{\pi^3(1-k_1^2)}}$$

$$\times [\,E(k_1) - (1-k_1^2)K(k_1)\,]\,[\,2E(k_2) - K(k_2)\,]. \qquad (2)$$

$$\int_0^\infty e^{-px^2} J_1(ax^2) J_1(bx^2) dx = \frac{2\sqrt{\gamma}}{3k_1k_2\sqrt{\pi^3(1-k_1^2)(1-k_2^2)}}$$

$$\times [\,(1-2k_1^2)E(k_1) - (1-k_1^2)K(k_1)\,]$$

$$\times [\,(1-2k_2^2)E(k_2) - (1-k_2^2)K(k_2)\,]. \qquad (3)$$

$$\int_0^\infty e^{-px^2} J_2(ax^2) J_0(bx^2) dx = \frac{2\sqrt{\gamma}}{9k_1^2(1-k_1^2)\sqrt{\pi^3}}$$

$$\times [\,(1-k_1^2)(2-3k_1^2)K(k_1) - 2(1-2k_1^2)E(k_1)\,]$$

$$\times [\,8(1-2k_2)E(k_2) - (5-8k_2^2)K(k_2)\,]. \qquad (4)$$

$$\int_0^\infty e^{-px^2} J_2(ax^2) J_1(bx^2) dx = \frac{2\sqrt{\gamma}}{15k_1^2k_2(1-k_1^2)\sqrt{\pi^3(1-k_2^2)}}$$

$$\times [2(1-k_1^2+k_1^4)E(k_1) - (1-k_1^2)(2-k_1^2)K(k_1)]$$

$$\times [(1-k_2^2)(1-8k_2^2)K(k_2) - (1-16k_2^2+16k_2^4)E(k_2)]. \qquad (5)$$

$$\int_0^\infty e^{-px^2} J_2(ax^2) J_2(bx^2) dx = \frac{2\sqrt{\gamma}}{105k_1^2k_2^2(1-k_1^2)(1-k_2^2)\sqrt{\pi^3}}$$

$$\times [(1-k_1^2)(2+5k_1^2-8k_1^4)K(k_1) - 2(1-2k_1^2)(1+4k_1^2-4k_1^4)E(k_1)]$$

$$\times [(1-k_2^2)(2+5k_2^2-8k_2^4)K(k_2) - 2(1-2k_2^2)(1+4k_2^2-4k_2^4)E(k_2)]. \qquad (6)$$

2.3. $k^2=\dfrac{a^2}{p^2+a^2}$

$$\int_0^\infty e^{-2px}J_0^2(ax)x^2dx=\frac{k}{4\pi p^2a}\left[\,(3-2k^2)E(k)-(1-k^2)K(k)\,\right]. \tag{1}$$

$$\int_0^\infty e^{-2px}J_1^2(ax)x^2dx=\frac{k}{4\pi p^2a}\left[\,(1-k^2)K(k)-(1-2k^2)E(k)\,\right]. \quad \text{BY 251 (561.08)}^2 \tag{2}$$

$$\int_0^\infty e^{-2px}J_1^2(ax)x^{-2}dx=\frac{4a}{3\pi k^3}\left[\,(1-k^2)K(k)-(1-2k^2)E(k)\,\right]-p. \tag{3}$$

$$\int_0^\infty e^{-2px}J_{\frac{3}{2}}^2(ax)xdx=\frac{1}{2\pi pak^3}\left[\,(16-16k^2+k^4)E(k)-8(1-k^2)(2-k^2)K(k)\,\right]. \tag{4}$$

$$\int_0^\infty e^{-2px}J_{\frac{3}{2}}^2(ax)x^{-2}dx=\frac{4a}{15\pi k^5}\left[\,(1-k^2)(4+3k^2)K(k)-(4+k^2-6k^4)E(k)\,\right]-\frac{p}{2}. \tag{5}$$

2.4. $k^2=\dfrac{a^2}{p^2+a^2}$

$$\int_0^\infty e^{-2px}J_2(ax)J_1(ax)x^{-1}dx=\frac{2}{3\pi k^3}\left[4(1-k^2)K(k)-(4-5k^2)E(k)\right]-\frac{p}{a}. \tag{1}$$

$$\int_0^\infty e^{-2px}J_3(ax)J_1(ax)dx=\frac{1}{3\pi ak^3}\left[(32-38k^2+3k^4)K(k)-2(16-23k^2)E(k)\right]-\frac{4p}{a^2}. \tag{2}$$

$$\int_0^\infty e^{-2px}J_3(ax)J_2(ax)x^{-1}dx=\frac{2}{15\pi k^5}\left[4(1-k^2)(8+k^2)K(k)-(32-12k^2-23k^4)E(k)\right]-\frac{p}{a}. \tag{3}$$

2.5. $k^2=\dfrac{4ab}{p^2+(a+b)^2}$

$$\int_0^\infty e^{-px}J_0(ax)J_0(bx)dx=\frac{k}{\pi\sqrt{ab}}K(k). \quad \text{BY 248 (560.01)} \tag{1}$$

$$\int_0^\infty e^{-px}J_0(ax)J_0(bx)xdx=\frac{pk^3}{4\pi(1-k^2)\sqrt{(ab)^3}}E(k). \quad \text{LU 316 (19)} \tag{2}$$

[2] The right-hand side in BY 251 (561.08) is incorrect.

$$\int_0^\infty e^{-px} J_1(ax)\, J_0(bx)\, x\,dx = \frac{k}{8\pi(1-k^2)a\sqrt{(ab)^3}}\left[k^2(a^2-b^2-p^2)E(k)+4ab(1-k^2)K(k)\right].$$

LU 317 (21) (3)

$$\int_0^\infty e^{-px} J_1(ax)\, J_1(bx)\, dx = \frac{1}{\pi k\sqrt{ab}}\left[(2-k^2)K(k)-2E(k)\right].$$

BY 249 (560.02) (4)

$$\int_0^\infty e^{-px} J_1(ax)\, J_1(bx)\, x\,dx = \frac{pk}{4\pi(1-k^2)\sqrt{(ab)^3}}\left[(2-k^2)E(k)-2(1-k^2)K(k)\right].$$

LU 316 (20) (5)

$$\int_0^\infty e^{-px} J_2(ax)\, J_2(bx)\, dx = \frac{1}{3\pi k^3\sqrt{ab}}\left[(4-k^2)(4-3k^2)K(k)-8(2-k^2)E(k)\right].$$

BY 248 (560.03) (6)

$$\int_0^\infty e^{-px} J_3(ax) J_3(bx)\, dx = \frac{1}{15\pi k^5\sqrt{ab}}\left[(128-128k^2+15k^4)\right.$$
$$\left.\times(2-k^2)K(k)-2(128-128k^2+23k^4)E(k)\right]. \quad (7)$$

$$\int_0^\infty e^{-px} J_4(ax) J_4(bx)\, dx = \frac{1}{105\pi k^7\sqrt{ab}}\left[(6144-12288k^2\right.$$
$$\left.+8000k^4-1856k^6+105k^8)K(k)-32(2-k^2)\,(96-96k^2+11k^4)E(k)\right]. \quad (8)$$

2.6 $\displaystyle k^2=\frac{4ab}{p^2+(a+b)^2}=\sin^2\alpha, \qquad \sin\beta=\frac{p}{\sqrt{p^2+(a-b)^2}}$

$\Lambda_0(\alpha,\beta)$ is Heuman's Lambda function.[3]

$$\int_0^\infty e^{-px} J_1(ax) J_0(bx)\, dx = -\frac{pk}{2\pi a\sqrt{ab}}K(k)-\frac{1}{2a}\Lambda_0(\alpha,\beta)+\frac{1}{a}, \qquad a>b;$$

$$=-\frac{pk}{2\pi a^2}K(k)+\frac{1}{2a}, \qquad a=b;$$

$$=-\frac{pk}{2\pi a\sqrt{ab}}K(k)+\frac{1}{2a}\Lambda_0(\alpha,\beta), \qquad a<b.$$

LU 317 (22) (1)

$$\int_0^\infty e^{-px} J_1(ax) J_0(bx)\, x^{-1}dx = \frac{2\sqrt{ab}}{\pi ak}E(k)+\frac{k(a^2-b^2)}{2\pi a\sqrt{ab}}K(k)+\frac{p}{2a}\Lambda_0(\alpha,\beta)-\frac{p}{a}, \qquad a>b;$$

$$=\frac{2}{\pi k}E(k)-\frac{p}{2a}, \qquad a=b;$$

[3] In connection with the Heuman's Lambda function the reader should consult BY pp. 35–37.

$$= \frac{2\sqrt{ab}}{\pi ak} E(k) + \frac{k(a^2-b^2)}{2\pi a\sqrt{ab}} K(k) - \frac{p}{2a} \Lambda_0(\alpha, \beta), \qquad a < b.$$

LU 318 (24) (2)

$$\int_0^\infty e^{-px} J_1(ax) J_1(bx) x^{-1} dx = \frac{p}{\pi k\sqrt{ab}} E(k) - \frac{pk(p^2+2a^2+2b^2)}{4\pi\sqrt{(ab)^3}} K(k) + \frac{(a^2-b^2)}{4ab} \Lambda_0(\alpha, \beta)$$

$$+ \frac{b}{2a}, \qquad a > b;$$

$$= \frac{p}{\pi ak} E(k) - \frac{pk(p^2+4a^2)}{4\pi a^3} K(k) + \frac{1}{2}, \qquad a = b;$$

$$= \frac{p}{\pi k\sqrt{ab}} E(k) - \frac{pk(p^2+2a^2+2b^2)}{4\pi\sqrt{(ab)^3}} K(k) - \frac{(a^2-b^2)}{4ab} \Lambda_0(\alpha, \beta)$$

$$+ \frac{a}{2b}, \qquad a < b.$$

LU 318 (25) (3)

$$\int_0^\infty e^{-px} J_2(ax) J_0(bx) dx = \frac{4\sqrt{ab}}{\pi a^2 k} E(k) - \frac{kb^2}{\pi a^2\sqrt{ab}} K(k) + \frac{p}{a^2} \Lambda_0(\alpha, \beta) - \frac{2p}{a^2}, \qquad a > b;$$

$$= \frac{4}{\pi ak} E(k) - \frac{k}{\pi a} K(k) - \frac{p}{a^2}, \qquad a = b;$$

$$= \frac{4\sqrt{ab}}{\pi a^2 k} E(k) - \frac{kb^2}{\pi a^2\sqrt{ab}} K(k) - \frac{p}{a^2} \Lambda_0(\alpha, \beta), \qquad a < b. \tag{4}$$

$$\int_0^\infty e^{-px} J_2(ax) J_0(bx) x dx = -\frac{pk}{\pi a^2\sqrt{ab}} K(k) - \frac{pk^3}{4\pi(1-k^2)\sqrt{(ab)^3}} E(k) - \frac{1}{a^2} \Lambda_0(\alpha, \beta)$$

$$+ \frac{2}{a^2}, \qquad a > b;$$

$$= -\frac{pk}{\pi a^3} K(k) - \frac{pk^3}{4\pi(1-k^2)a^3} E(k) + \frac{1}{a^2}, \qquad a = b;$$

$$= -\frac{pk}{\pi a^2\sqrt{ab}} K(k) - \frac{pk^3}{4\pi(1-k^2)\sqrt{(ab)^3}} E(k)$$

$$+ \frac{1}{a^2} \Lambda_0(\alpha, \beta), \qquad a < b. \tag{5}$$

$$\int_0^\infty e^{-pz} J_2(ax) J_1(bx)\, dx = \frac{2p}{\pi k a \sqrt{ab}} E(k) - \frac{pk(p^2+a^2+2b^2)}{2\pi a \sqrt{(ab)^3}} K(k) - \frac{b}{2a^2} \Lambda_0(\alpha, \beta)$$

$$+ \frac{b}{a^2}, \qquad a > b;$$

$$= \frac{2p}{\pi a^2 k} E(k) - \frac{pk(p^2+3a^2)}{2\pi a^4} K(k) + \frac{1}{2a}, \qquad a = b;$$

$$= \frac{2p}{\pi k a \sqrt{ab}} E(k) - \frac{pk(p^2+a^2+2b^2)}{2\pi a \sqrt{(ab)^3}} K(k) + \frac{b}{2a^2} \Lambda_0(\alpha, \beta)$$

$$+ \frac{1}{b}, \qquad a < b. \tag{6}$$

2.7. $k^2 = \dfrac{4ab}{(a+b)^2}$

$$\int_0^\infty J_1(ax) J_1(bx)\, x^{-2} dx = \frac{a+b}{3\pi ab} [(a^2+b^2)E(k) - (a-b)^2 K(k)]. \tag{1}$$

$$\int_0^\infty J_2(ax) J_1(bx)\, x^{-1} dx = \frac{1}{3\pi a^2 b} [(a-b)(a^2+2b^2)K(k) - (a+b)(a^2-2b^2)E(k)]. \tag{2}$$

$$\int_0^\infty J_3(ax) J_1(bx)\, dx = \frac{1}{3\pi a^3 b(a+b)} [(a^4+a^2b^2-8b^4)K(k) - (a+b)^2(a^2-8b^2)E(k)]. \tag{3}$$

$$\int_0^\infty J_3(ax) J_2(bx)\, x^{-1} dx = \frac{1}{15\pi a^3 b^2} [(a-b)(2a^4+5a^2b^2+8b^4)K(k)$$

$$- (a+b)(2a^4+3a^2b^2-8b^4)E(k)]. \tag{4}$$

$$\int_0^\infty J_4(ax) J_2(bx)\, dx = \frac{2}{15\pi a^4 b^2 (a+b)} [(a^6+4a^4b^2+4a^2b^4-24b^6)$$

$$\times K(k) - (a+b)^2(a^4+4a^2b^2-24b^4)E(k)]. \tag{5}$$

2.8. $k^2 = \dfrac{b-\sqrt{b^2-a^2}}{2b}, \qquad b > a$

$$\int_0^\infty J_0^2(ax) J_0(2bx)\, dx = \frac{2}{\pi^2 b} [K(k)]^2. \tag{1}$$

$$\int_0^\infty J_0^2(ax) J_2(2bx) dx = \frac{2}{\pi^2 b} [2E(k) - K(k)]^2. \tag{2}$$

$$\int_0^\infty J_1^2(ax) J_0(2bx) dx = -\frac{8b}{\pi^2 a^2} [E(k) - (1-k^2)K(k)]^2. \tag{3}$$

$$\int_0^\infty J_1^2(ax) J_2(2bx) dx = \frac{8b}{3\pi^2 a^2} [(1-k^2)K(k) - (1-2k^2)E(k)]^2. \tag{4}$$

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$$\int_0^\infty J_{\frac{3}{2}}^2(ax)J_0(2bx)dx=\frac{32b^3}{9\pi^2a^4}[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)]^2. \tag{5}$$

$$\int_0^\infty J_{\frac{3}{2}}^2(ax)J_2(2bx)dx=-\frac{32b^3}{15\pi^2a^4}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]^2. \tag{6}$$

$$\int_0^\infty J_{\frac{3}{2}}^2(ax)J_4(2bx)dx=\frac{32b^3}{105\pi^2a^4}[(2+5k^2-8k^4)(1-k^2)K(k) -2(1-2k^2)(1+4k^2-4k^4)E(k)]^2. \tag{7}$$

$$\int_0^\infty J_3^2(ax)J_0(2bx)dx=-\frac{128b^5}{225\pi^2a^6}[(8-23k^2+23k^4)E(k)-(1-k^2)(8-19k^2+15k^4)K(k)]^2. \tag{8}$$

$$\int_0^\infty J_3^2(ax)J_2(2bx)dx=\frac{128b^5}{315\pi^2a^6}[(8-15k^2+3k^4)(1-k^2)K(k)-(8-19k^2+9k^4-6k^6)E(k)]^2. \tag{9}$$

2.9. $k_1^2=\dfrac{abcd}{\Delta^2}$, $\quad k_2^2=\dfrac{\Delta^2}{abcd}$

$$16\Delta^2=(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)$$

$$\int_0^\infty J_0(ax)J_0(bx)J_0(cx)J_0(dx)xdx=\frac{1}{\pi^2\Delta}K(k_1), \qquad \Delta^2>abcd;$$

$$=\frac{1}{\pi^2\sqrt{abcd}}K(k_2), \qquad \Delta^2<abcd. \qquad \text{WA 414 (9)} \tag{1}$$

3. Integrands Involving Bessel Functions of the Second Kind

3.1. $k^2=\dfrac{\sqrt{p^2+a^2}+p}{2\sqrt{p^2+a^2}}$

$$\int_0^\infty e^{-px^2}Y_0(ax^2)dx=-\frac{\sqrt{2k^2-1}}{\sqrt{\pi p}}K(k). \tag{1}$$

$$\int_0^\infty e^{-px^2}Y_0(ax^2)x^2dx=\frac{\sqrt{(2k^2-1)^3}}{2\sqrt{\pi p^3}}[2E(k)-K(k)]. \tag{2}$$

$$\int_0^\infty e^{-px^2}Y_0(ax^2)x^4dx=-\frac{\sqrt{(2k^2-1)^5}}{4\sqrt{\pi p^5}}[8(1-2k^2)E(k)-(5-8k^2)K(k)]. \tag{3}$$

$$\int_0^\infty e^{-px^2}Y_0(ax^2)x^6dx=\frac{\sqrt{(2k^2-1)^7}}{8\sqrt{\pi p^7}}[2(23-128k^2+128k^4)E(k)-(31-144k^2+128k^4)K(k)]. \tag{4}$$

$$\int_0^\infty e^{-px^2}Y_1(ax^2)x^2dx=-\frac{\sqrt{(2k^2-1)^3}}{2k\sqrt{\pi p^3(1-k^2)}}[(1-k^2)K(k)-(1-2k^2)E(k)]. \tag{5}$$

$$\int_0^\infty e^{-px^2}Y_1(ax^2)x^4dx=\frac{\sqrt{(2k^2-1)^5}}{4k\sqrt{\pi p^5(1-k^2)}}$$
$$\times[(1-k^2)(1-8k^2)K(k)-(1-16k^2+16k^4)E(k)]. \tag{6}$$

$$\int_0^\infty e^{-px^2}Y_1(ax^2)x^6dx=-\frac{\sqrt{(2k^2-1)^7}}{8k\sqrt{\pi p^7(1-k^2)}}$$
$$\times[(3-80k^2+128k^4)(1-k^2)K(k)-(3-134k^2+384k^4-256k^6)E(k)]. \tag{7}$$

$$\int_0^\infty e^{-px^2}Y_2(ax^2)x^4dx=-\frac{\sqrt{(2k^2-1)^5}}{4k^2(1-k^2)\sqrt{\pi p^5}}$$
$$\times[(2+5k^2-8k^4)(1-k^2)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \tag{8}$$

$$\int_0^\infty e^{-px^2}Y_2(ax^2)x^6dx=\frac{\sqrt{(2k^2-1)^7}}{8k^2(1-k^2)\sqrt{\pi p^7}}$$
$$\times[(2+15k^2-144k^4+128k^6)(1-k^2)K(k)-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]. \tag{9}$$

3.2. $k^2=\dfrac{p^2}{p^2+a^2}$

$$\int_0^\infty e^{-2px}Y_0(ax)J_0(ax)dx=-\frac{k}{\pi p}K(k)\cdot \qquad \text{OL 122 (13.18)} \tag{1}$$

$$\int_0^\infty e^{-2px}Y_0(ax)J_0(ax)xdx=-\frac{k}{2\pi p^2}[K(k)-E(k)]. \tag{2}$$

$$\int_0^\infty e^{-2px}Y_0(ax)J_0(ax)x^2dx=-\frac{k}{4\pi p^3}[(1+k^2)K(k)-(1+2k^2)E(k)]. \tag{3}$$

$$\int_0^\infty e^{-2px}Y_0(ax)J_0(ax)x^3dx=-\frac{k}{8\pi p^4}[2(1-2k^4)K(k)+(2+k^2+8k^4)E(k)]. \tag{4}$$

3.3. $k^2=\dfrac{a-\sqrt{a^2-b^2}}{2a}, \qquad a>b$

$$\int_0^\infty Y_0(ax^2)J_0(bx^2)dx=-\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^2\sqrt{a}}K(k). \tag{1}$$

$$\int_0^\infty Y_0(ax^2)J_0(bx^2)x^2dx=\frac{\Gamma^2\left(\frac{3}{4}\right)}{\pi^2\sqrt{a(a^2-b^2)}}K(k). \tag{2}$$

$$\int_0^\infty Y_0(ax^2)J_1(bx^2)dx=\frac{4\Gamma^2\left(\frac{3}{4}\right)\sqrt{a}}{\pi^2 b}\left[E(k)-(1-k^2)K(k)\right]. \tag{3}$$

$$\int_0^\infty Y_0(ax^2)J_2(bx^2)dx=-\frac{2\Gamma^2\left(\frac{1}{4}\right)\sqrt{a^3}}{9\pi^2 b^2}\left[(2-3k^2)(1-k^2)K(k)-2(1-2k^2)E(k)\right]. \tag{4}$$

$$\int_0^\infty Y_1(ax^2)J_1(bx^2)\,dx=-\frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{a}}{3\pi^2 b}\left[(1-k^2)K(k)-(1-2k^2)E(k)\right]. \tag{5}$$

$$\int_0^\infty Y_1(ax^2)J_1(bx^2)\,x^2dx=\frac{2\Gamma^2\left(\frac{3}{4}\right)\sqrt{a}}{\pi^2 b\sqrt{a^2-b^2}}\left[(1-k^2)K(k)-(1-2k^2)E(k)\right]. \tag{6}$$

$$\int_0^\infty Y_1(ax^2)J_2(bx^2)\,dx=\frac{8\Gamma^2\left(\frac{3}{4}\right)\sqrt{a^3}}{5\pi^2 b^2}\left[2(1-k^2+k^4)E(k)-(2-k^2)(1-k^2)K(k)\right]. \tag{7}$$

$$\int_0^\infty Y_2(ax^2)J_2(bx^2)\,dx=-\frac{2\Gamma^2\left(\frac{1}{4}\right)\sqrt{a^3}}{21\pi^2 b^2}\left[(2+5k^2-8k^4)\ (1-k^2)K(k)\right.$$
$$\left.-2(1-2k^2)(1+4k^2-4k^4)E(k)\right]. \tag{8}$$

3.4. $k^2=\dfrac{b^2}{a^2+b^2}$

$$\int_0^\infty Y_0(ax)K_0(bx)\,dx=-\frac{k}{b}K(k). \qquad \text{OB 152 (2.29)} \tag{1}$$

4. Integrands Involving Modified Bessel Functions of the First Kind [4]

4.1. $k^2=\dfrac{2a}{p+a}$, $p>a$

$$\int_0^\infty e^{-px^2}I_0(ax^2)dx=\frac{k}{\sqrt{2\pi a}}K(k). \qquad \text{OL 119 (13.5)} \tag{1}$$

$$\int_0^\infty e^{-px^2}I_0(ax^2)x^2dx=\frac{k^3}{4(1-k^2)\sqrt{2\pi a^3}}E(k). \qquad \text{NM 151} \tag{2}$$

$$\int_0^\infty e^{-px^2}I_0(ax^2)x^4dx=\frac{k^5}{16(1-k^2)^2\sqrt{2\pi a^5}}\left[2(2-k^2)E(k)-(1-k^2)K(k).\right] \qquad \text{NM 151} \tag{3}$$

[4] The integrals of the same types as compiled in 2.1–2.6, 6.1–6.7 which involve $I_n(x)$ in place of $J_n(x)$ are also expressible in terms of $K(k)$ and $E(k)$, whose results may be deduced from those without difficulty by using the relation $J_n(ix)=i^nI_n(x)$. Some of those results are also found in: WA 391 (5); OL 122 (13.21)–(13.23), 123 (13.24)–(13.27); OB 152 (2.30), (2.31), 153 (2.38), 154 (2.40), (2.41); BY 251 (562.01)–(562.04).

$$\int_0^\infty e^{-px^2} I_0(ax^2) x^6 dx = \frac{k^7}{64(1-k^2)^3\sqrt{2\pi a^7}} [(23-23k^2+8k^4)E(k) - 4(1-k^2)(2-k^2)K(k)]. \tag{4}$$

$$\int_0^\infty e^{-px^2} I_1(ax^2) dx = \frac{1}{k\sqrt{2\pi a}} [(2-k^2)K(k) - 2E(k)]. \tag{5}$$

$$\int_0^\infty e^{-px^2} I_1(ax^2)\, x^2 dx = \frac{k}{4(1-k^2)\sqrt{2\pi a^3}} [(2-k^2)E(k) - 2(1-k^2)K(k)]. \tag{6}$$

$$\int_0^\infty e^{-px^2} I_1(ax^2)\, x^4 dx = \frac{k^3}{16(1-k^2)^2\sqrt{2\pi a^5}} [2(1-k^2+k^4)E(k) - (2-k^2)(1-k^2)K(k)].$$

NM 151 (7)

$$\int_0^\infty e^{-px^2} I_1(ax^2)\, x^{-2} dx = \frac{2\sqrt{2a}}{3\sqrt{\pi k^3}} [(2-k^2)E(k) - 2(1-k^2)K(k)]. \tag{8}$$

$$\int_0^\infty e^{-px^2} I_2(ax^2)\, dx = \frac{1}{3k^3\sqrt{2\pi a}} [(16-16k^2+3k^4)K(k) - 8(2-k^2)(1-k^2)E(k)]. \tag{9}$$

$$\int_0^\infty e^{-px^2} I_2(ax^2) x^2 dx = \frac{1}{4k(1-k^2)\sqrt{2\pi a^3}} [(16-16k^2+k^4)E(k) - 8(1-k^2)(2-k^2)K(k)]. \tag{10}$$

$$\int_0^\infty e^{-px^2} I_2(ax^2)\, x^4 dx = \frac{k}{16(1-k^2)^2\sqrt{2\pi a^5}} [(16-16k^2-k^4) \times (1-k^2)K(k) - 2(2-k^2)(4-4k^2-k^4)E(k)]. \tag{11}$$

$$\int_0^\infty e^{-px^2} I_2(ax^2) x^6 dx = \frac{k^3}{64(1-k^2)^3\sqrt{2\pi a^7}} [4(2-2k^2-k^4) \times (2-k^2)(1-k^2)K(k) - (16-32k^2+9k^4+7k^6-8k^8)E(k)]. \tag{12}$$

$$\int_0^\infty e^{-px^2} I_2(ax^2) x^{-2} dx = \frac{2\sqrt{2a}}{15\sqrt{\pi k^5}} [(16-16k^2+k^4)E(k) - 8(2-k^2)(1-k^2)K(k)]. \tag{13}$$

$$\int_0^\infty e^{-px^2} I_3(ax^2) dx = \frac{1}{15k^5\sqrt{2\pi a}} [(128-128k^2+15k^4) \times (2-k^2)K(k) - 2(128-128k^2+23k^4)E(k)]. \tag{14}$$

$$\int_0^\infty e^{-px^2} I_3(ax^2) x^2 dx = \frac{1}{12k^3(1-k^2)\sqrt{2\pi a^3}} [(128-128k^2+3k^4) \times (2-k^2)E(k) - 2(1-k^2)(128-128k^2+27k^4)K(k)]. \tag{15}$$

$$\int_0^\infty e^{-px^2} I_3(ax^2) x^4 dx = \frac{1}{16k(1-k^2)^2\sqrt{2\pi a^5}} [(128-128k^2-k^4) \times (2-k^2)(1-k^2)K(k) - 2(128-256k^2+135k^4-7k^6-k^8)E(k)]. \tag{16}$$

$$\int_0^\infty e^{-px^2} I_3(ax^2) x^6 dx = \frac{k}{64(1-k^2)^3\sqrt{2\pi a^7}} [(128-256k^2+99k^4+29k^6+8k^8)(2-k^2)E(k) - 2(1-k^2)(128-256k^2+123k^4+5k^6+2k^8)K(k)]. \quad (17)$$

$$\int_0^\infty e^{-px^2} I_3(ax^2)\, x^{-2} dx = \frac{2\sqrt{2a}}{105\sqrt{\pi k^7}} [(128-128k^2+3k^4) \times (2-k^2)E(k) - 2(1-k^2)(128-128k^2+27k^4)K(k)]. \quad (18)$$

$$\int_0^\infty e^{-px^2} I_3(ax^2)\, x^{-4} dx = \frac{8\sqrt{2a^3}}{945\sqrt{\pi k^9}} [(128-128k^2-k^4)(1-k^2) \times (2-k^2)K(k) - 2(128-256k^2+135k^4-7k^6-k^8)E(k)]. \quad (19)$$

$$\int_0^\infty e^{-px^2} I_3(ax^2)\, x^{-6} dx = \frac{32\sqrt{2a^5}}{10395\sqrt{\pi k^{11}}} [(128-256k^2+99k^4+29k^6+8k^8) \times (2-k^2)E(k) - 2(1-k^2)(128-256k^2+123k^4+5k^6+2k^8)K(k)]. \quad (20)$$

4.2 $k^2 = \frac{1}{b^2}(\sqrt{a+b}-\sqrt{a})^2(\sqrt{a+2b}-\sqrt{a+b})^2, \quad k^2+k'^2=1$

$$\int_0^\infty e^{-2(a+b)x} I_0^2(ax) I_0(2bx)\, dx = \frac{4}{\pi^2 b\sqrt{a}}(\sqrt{a+2b}-\sqrt{a+b})\ K(k)K(k'). \quad \text{MV 228} \quad (1)$$

4.3. $k^2 = \left(\frac{p-a}{p+a}\right)^2$

$$\int_0^\infty e^{-2px} I_0(ax) K_0(ax)\, dx = \frac{1}{p+a} K(k). \quad \text{ET II 370 (48)} \quad (1)$$

$$\int_0^\infty e^{-2px} I_0(ax) K_0(ax) x dx = \frac{1}{4p(a^2-p^2)} [(p+a)E(k) - 2pK(k)]. \quad (2)$$

$$\int_0^\infty e^{-2px} I_0(ax) K_0(ax)\, x^2 dx = \frac{1}{8p^2(a+p)(a-p)^2} [(a^2-3p^2)E(k) + 2p(2p-a)K(k)]. \quad (3)$$

4.4. $k^2 = \left(\frac{b}{\sqrt{a^2+b^2}+a}\right)^2, \quad k^2+k'^2=1$

$$\int_0^\infty I_0(ax) K_0(ax) J_0(2bx) dx = \frac{k'^2}{\pi b} K(k)K(k'). \quad \text{OB 23 (2.116)} \quad (1)$$

$$\int_0^\infty I_0(ax) K_0(ax) J_2(2bx) dx = \frac{1}{\pi b k'^2}[2E(k) - (1-k^2)K(k)][(2-k'^2)K(k') - 2E(k')]. \quad (2)$$

$$\int_0^\infty I_1(ax) K_1(ax) J_0(2bx) dx = \frac{k'^2}{\pi b k^2} E(k')[K(k) - E(k)]. \quad (3)$$

$$\int_0^\infty I_1(ax)K_1(ax)J_2(2bx)dx=\frac{1}{3\pi bk^2k'^2}[(1+k^2)E(k)$$
$$-(1-k^2)K(k)][(2-k'^2)E(k')-2(1-k'^2)K(k')]. \quad (4)$$

$$\int_0^\infty I_2(ax)K_2(ax)J_0(2bx)dx=\frac{k'^2}{9\pi bk^4}[(2+k^2)K(k)$$
$$-2(1+k^2)E(k)][2(2-k'^2)E(k')-(1-k'^2)K(k')]. \quad (5)$$

5. Integrands Involving Modified Bessel Functions of the Second Kind

5.1.[5] $k_1^2=\frac{p-a}{p+a}$, $k_2^2=\frac{a-p}{2a}$

$$\int_0^\infty e^{-px^2}K_0(ax^2)\,dx=\frac{\sqrt{\pi}}{\sqrt{p+a}}K(k_1), \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{\sqrt{2a}}K(k_2), \qquad p<a. \quad (1)$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)\,x^2dx=\frac{\sqrt{\pi(p+a)}}{2(p^2-a^2)}[K(k_1)-E(k_1)], \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{2(a^2-p^2)\sqrt{2a}}[2aE(k_2)-(p+a)K(k_2)], \qquad p<a. \quad (2)$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)\,x^4dx=\frac{\sqrt{\pi(p+a)}}{4(p^2-a^2)^2}[(3p+a)K(k_1)-4pE(k_1)], \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{4(a^2-p^2)^2\sqrt{2a}}[(p+a)(3p+a)K(k_2)-8paE(k_2)], \qquad p<a. \quad (3)$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)x^6dx=\frac{\sqrt{\pi(p+a)}}{8(p^2-a^2)^3}[(15p^2+8pa+9a^2)K(k_1)-(23p^2+9a^2)E(k_1)], \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{8(a^2-p^2)^3\sqrt{2a}}[2a(23p^2+9a^2)E(k_2)-(p+a)(15p^2+8pa+9a^2)K(k_2)], \qquad p<a. \quad (4)$$

$$\int_0^\infty e^{-px^2}K_1(ax^2)x^2dx=\frac{\sqrt{\pi(p+a)}}{2a(p^2-a^2)}[pE(k_1)-aK(k_1)], \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{2(a^2-p^2)\sqrt{2a}}[(p+a)K(k_2)-2pE(k_2)], \qquad p<a. \quad (5)$$

$$\int_0^\infty e^{-px^2}K_1(ax^2)x^4dx=\frac{\sqrt{\pi(p+a)}}{4a(p^2-a^2)^2}[(p^2+3a^2)E(k_1)-a(p+3a)K(k_1)], \qquad p>a;$$
$$=\frac{\sqrt{\pi}}{4(a^2-p^2)^2\sqrt{2a}}[2(p^2+3a^2)E(k_2)-(p+a)(p+3a)K(k_2)], \qquad p<a. \quad (6)$$

[5] For negative p, the second results are available.

$$\int_0^\infty e^{-px^2}K_1(ax^2)x^6dx=\frac{\sqrt{\pi(p+a)}}{8a(p^2-a^2)^3}\,[p(3p^2+29a^2)E(k_1)$$

$$-a(3p^2+24pa+5a^2)K(k_1)\,],\qquad p>a;$$

$$=\frac{\sqrt{\pi}}{8(a^2-p^2)^3\sqrt{2a}}\,[\,(p+a)\,(3p^2+24pa+5a^2)K(k_2)$$

$$-2p(3p^2+29a^2)E(k_2)\,],\qquad p<a. \tag{7}$$

$$\int_0^\infty e^{-px^2}K_2(ax^2)x^4dx=\frac{\sqrt{\pi(p+a)}}{4a^2(p^2-a^2)^2}\,[a(5a^2+3pa-4p^2)K(k_1)$$

$$-4p(2a^2-p^2)E(k_1)\,],\qquad p>a;$$

$$=\frac{\sqrt{\pi}}{4a(a^2-p^2)^2\sqrt{2a}}\,[\,(5a^2+3pa-4p^2)$$

$$\times(p+a)K(k_2)-8p(2a^2-p^2)E(k_2)],\qquad p<a. \tag{8}$$

5.2. $k_1^2=\dfrac{a-\sqrt{a^2-b^2}}{2a}$, $\qquad k_2^2=\dfrac{b-\sqrt{b^2-a^2}}{2b}$

$$k_1^2+k_1'^2=1,\qquad k_2^2+k_2'^2=1$$

$$\int_0^\infty K_0(ax^2)K_0(bx^2)\,dx=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\,\sqrt{2a}}\,[K(k_1)+K(k_1')\,],\qquad a>b.\qquad \text{OB 153 (2.33)} \tag{1}$$

$$\int_0^\infty K_0(ax^2)K_0(bx^2)\,x^2dx=\frac{\Gamma^2\left(\frac{3}{4}\right)}{2\,\sqrt{2a^3}\,(k_1'^2-k_1^2)}\,[K(k_1')-K(k_1)\,],\qquad a>b. \tag{2}$$

$$\int_0^\infty K_1(ax^2)\;K_0(bx^2)\,x^2dx$$

$$=\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\,\sqrt{2a^3}\,(k_1'^2-k_1^2)}\,[2E(k_1)-K(k_1)+K(k_1')-2E(k_1')\,],\qquad a>b;$$

$$=\frac{a\Gamma^2\left(\frac{1}{4}\right)}{16\,\sqrt{2b^5}\,(k_2'^2-k_2^2)\,(k_2k_2')^2}\,[\,(1-k_2^2)K(k_2)-E(k_2)+E(k_2')-(1-k_2'^2)K(k_2')\,],\qquad a<b. \tag{3}$$

5.3 $k^2=\left(\dfrac{a-b}{a+b}\right)^2$

$$\int_0^\infty K_0(ax)K_0(bx)dx=\frac{\pi}{a+b}\,K(k). \tag{1}$$

$$\int_0^\infty K_0(ax)K_0(bx)x^2dx=\frac{\pi}{(a+b)(a-b)^2}\,[K(k)-E(k)]. \tag{2}$$

$$\int_0^\infty K_1(ax)K_0(bx)xdx=\frac{\pi}{2a(a^2-b^2)}\,[2aK(k)-(a+b)E(k)]. \tag{3}$$

$$\int_0^\infty K_1(ax)K_0(bx)x^3dx=\frac{\pi}{2a(a+b)^2(a-b)^3}\,[2a(3a+b)K(k)-(7a^2+b^2)E(k)]. \tag{4}$$

$$\int_0^\infty K_1(ax)K_1(bx)x^2dx=\frac{\pi}{2ab(a+b)(a-b)^2}\,[(a^2+b^2)E(k)-2abK(k)]. \tag{5}$$

$$\int_0^\infty K_2(ax)K_0(bx)x^2dx=\frac{\pi}{a^2(a+b)(a-b)^2}\,[a(3a-2b)K(k)-(2a^2-b^2)E(k)]. \tag{6}$$

$$\int_0^\infty K_2(ax)K_1(bx)x^3dx=\frac{\pi}{2a^2b(a+b)^2(a-b)^3}\,[(3a^4+7a^2b^2-2b^4)E(k)-2ab(3a^2+3ab-2b^2)K(k)]. \tag{7}$$

5.4 $\quad k_1^2=\dfrac{a-\sqrt{a^2-b^2}}{2a}, \qquad k_2^2=\dfrac{b-\sqrt{b^2-a^2}}{2b}$

$$k_1^2+k_1'^2=1, \qquad k_2^2+k_2'^2=1$$

$$\int_0^\infty K_0^2(ax)K_0(2bx)dx=\frac{\pi}{2a}K(k_1)K(k_1'), \qquad a>b;$$

$$=\frac{\pi}{4b}\,[K^2(k_2)+K^2(k_2')], \qquad a<b. \qquad \text{OB 154 (2.39)} \tag{1}$$

6. Integrands Involving Products of Bessel and Modified Bessel Functions

6.1 $\quad k_1^2=\dfrac{2}{1+\sqrt{1+a^2\gamma^2}}, \qquad k_2^2=\dfrac{1-\sqrt{1-b^2\gamma^2}}{2}$

$$\gamma^2=\frac{2}{p^2-a^2+b^2+\sqrt{(p^2-a^2+b^2)^2+4a^2b^2}}$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)J_0(bx^2)dx=\frac{k_1\sqrt{\gamma}}{\sqrt{\pi}}K(k_1)K(k_2). \tag{1}$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)J_1(bx^2)dx=\frac{\sqrt{\gamma}}{k_1k_2\sqrt{\pi(1-k_2^2)}}\,[(2-k_1^2)K(k_1)-2E(k_1)][E(k_2)-(1-k_2^2)K(k_2)]. \tag{2}$$

$$\int_0^\infty e^{-px^2}K_0(ax^2)J_2(bx^2)dx=\frac{\sqrt{\gamma}}{9\sqrt{\pi}k_1^3k_2^2(1-k_2^2)}$$
$$\times[(16-16k_1^2+3k_1^4)K(k_1)-8(2-k_1^2)E(k_1)]$$
$$\times[(1-k_2^2)(2-3k_2^2)K(k_2)-2(1-2k_2^2)E(k_2)]. \qquad (3)$$

$$\int_0^\infty e^{-px^2}K_1(ax^2)J_1(bx^2)dx=\frac{\sqrt{\gamma}}{3k_1k_2\sqrt{\pi(1-k_1^2)(1-k_2^2)}}$$
$$\times[(2-k_1^2)E(k_1)-2(1-k_1^2)K(k_1)][(1-k_2^2)K(k_2)-(1-2k_2^2)E(k_2)]. \qquad (4)$$

$$\int_0^\infty e^{-px^2}K_1(ax^2)J_2(bx^2)dx=\frac{\sqrt{\gamma}}{15\sqrt{\pi}k_1^3k_2^2(1-k_2^2)\sqrt{1-k_1^2}}$$
$$\times[(16-16k_1^2+k_1^4)E(k_1)-8(1-k_1^2)(2-k_1^2)K(k_1)]$$
$$\times[2(1-k_2^2+k_2^4)E(k_2)-(1-k_2^2)(2-k_2^2)K(k_2)]. \qquad (5)$$

$$\int_0^\infty e^{-px^2}K_2(ax^2)J_2(bx^2)dx=\frac{\sqrt{\gamma}}{105\sqrt{\pi}k_1^3k_2^2(1-k_1^2)(1-k_2^2)}$$
$$\times[(1-k_1^2)(16-16k_1^2-k_1^4)K(k_1)-2(2-k_1^2)(4-4k_1^2-k_1^4)E(k_1)]$$
$$\times[(1-k_2^2)(2+5k_2^2-8k_2^4)K(k_2)-2(1-2k_2^2)(1+4k_2^2-4k_2^4)E(k_2)]. \qquad (6)$$

6.2. $k^2=\left(\frac{b}{\sqrt{a^2+b^2}+a}\right)^2$

$$\int_0^\infty K_0(ax^2)J_0(bx^2)dx=\frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{k}}{2\pi\sqrt{b}}K(k). \qquad \text{OB 22 (2.108)} \qquad (1)$$

$$\int_0^\infty K_0(ax^2)J_0(bx^2)x^2dx=\frac{\Gamma^2\left(\frac{3}{4}\right)\sqrt{k}}{\pi\sqrt{b(a^2+b^2)}}K(k). \qquad \text{OB 21 (2.107)} \qquad (2)$$

$$\int_0^\infty K_0(ax^2)J_1(bx^2)dx=\frac{2\Gamma^2\left(\frac{3}{4}\right)}{\pi\sqrt{bk}}[K(k)-E(k)]. \qquad (3)$$

$$\int_0^\infty K_0(ax^2)J_2(bx^2)dx=\frac{\Gamma^2\left(\frac{1}{4}\right)}{18\pi\sqrt{bk^3}}[(2+k^2)K(k)-2(1+k^2)E(k)]. \qquad (4)$$

$$\int_0^\infty K_1(ax^2)J_1(bx^2)dx=\frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{b}}{12\pi a\sqrt{k^3}}[(1+k^2)E(k)-(1-k^2)K(k)]. \qquad (5)$$

$$\int_0^\infty K_1(ax^2)J_1(bx^2)x^2dx=\frac{\Gamma^2\left(\frac{3}{4}\right)\sqrt{b}}{2\pi a\sqrt{(a^2+b^2)k^3}}[(1+k^2)E(k)-(1-k^2)K(k)]. \qquad (6)$$

$$\int_0^\infty K_1(ax^2)J_2(bx^2)\,dx=\frac{\Gamma^2\left(\frac{3}{4}\right)\sqrt{b}}{5\pi a\sqrt{k^5}}\left[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)\right]. \tag{7}$$

$$\int_0^\infty K_2(ax^2)J_2(bx^2)\,dx=\frac{\Gamma^2\left(\frac{1}{4}\right)\sqrt{b^3}}{168\pi a^2\sqrt{k^7}}\left[(2-9k^2-k^4)\right.$$
$$\left.\times(1-k^2)K(k)-2(1+k^2)(1-6k^2+k^4)E(k)\right]. \tag{8}$$

6.3. $k^2=\frac{b^2}{a^2+b^2}$

$$\int_0^\infty K_0(ax)J_0(bx)dx=\frac{k}{b}K(k). \qquad \text{OB 22 (2.109), 152 (2.28)} \tag{1}$$

$$\int_0^\infty K_0(ax)J_0(bx)\,x^2dx=\frac{k^3}{b^3}\left[2E(k)-K(k)\right]. \tag{2}$$

$$\int_0^\infty K_0(ax)J_0(bx)\,x^4dx=\frac{3k^5}{b^5}\left[8(1-2k^2)E(k)-(5-8k^2)K(k)\right]. \tag{3}$$

$$\int_0^\infty K_0(ax)J_0(bx)x^6dx=\frac{15k^7}{b^7}\left[2(23-128k^2+128k^4)E(k)-(31-144k^2+128k^4)K(k)\right]. \tag{4}$$

$$\int_0^\infty K_0(ax)J_1(bx)\,x\,dx=\frac{k}{b^2}\left[K(k)-E(k)\right]. \tag{5}$$

$$\int_0^\infty K_0(ax)J_1(bx)\,x^3dx=\frac{k^3}{b^4}\left[(1-4k^2)K(k)-(1-8k^2)E(k)\right]. \tag{6}$$

$$\int_0^\infty K_0(ax)J_1(bx)\,x^5dx=\frac{3k^5}{b^6}\left[(3-4k^2)(1-16k^2)K(k)-(3-88k^2+128k^4)E(k)\right]. \tag{7}$$

$$\int_0^\infty K_0(ax)J_1(bx)\,x^{-1}dx=\frac{1}{k}\left[K(k)-E(k)\right]. \tag{8}$$

$$\int_0^\infty K_0(ax)J_2(bx)\,dx=\frac{1}{bk}\left[(2-k^2)K(k)-2E(k)\right]. \tag{9}$$

$$\int_0^\infty K_0(ax)J_2(bx)\,x^2dx=\frac{k}{b^3}\left[(2+k^2)K(k)-2(1+k^2)E(k)\right]. \tag{10}$$

$$\int_0^\infty K_0(ax)J_2(bx)\,x^4dx=\frac{k^3}{b^5}\left[(2+7k^2-24k^4)K(k)-2(1+4k^2-24k^4)E(k)\right]. \tag{11}$$

$$\int_0^\infty K_0(ax)J_2(bx)\,x^{-2}dx=\frac{b}{9k^3}\left[(2+k^2)K(k)-2(1+k^2)E(k)\right]. \tag{12}$$

$$\int_0^\infty K_0(ax)J_3(bx)\,x\,dx=\frac{1}{b^2k}\left[(8-5k^2)K(k)-(8-k^2)E(k)\right]. \tag{13}$$

$$\int_0^\infty K_0(ax)J_3(bx)\,x^3dx=\frac{k}{b^4}\left[(8+3k^2+4k^4)K(k)-(8+7k^2+8k^4)\,E(k)\right]. \tag{14}$$

$$\int_0^\infty K_0(ax)J_3(bx)\ x^5dx=\frac{k^3}{b^6}\left[(8+19k^2+60k^4-192k^6)K(k)-(8+23k^2+72k^4-384k^6)E(k)\right]. \quad (15)$$

$$\int_0^\infty K_0(ax)J_3(bx)\ x^{-1}dx=\frac{1}{9k^3}\left[(8-5k^2)K(k)-(8-k^2)E(k)\right]. \quad (16)$$

$$\int_0^\infty K_0(ax)J_3(bx)\ x^{-3}dx=\frac{b^2}{225k^5}\left[(8+3k^2+4k^4)K(k)-(8+7k^2+8k^4)E(k)\right]. \quad (17)$$

$$\int_0^\infty K_1(ax)J_0(bx)\ x\,dx=\frac{k}{ab}E(k). \quad (18)$$

$$\int_0^\infty K_1(ax)J_0(bx)\ x^3dx=\frac{k^3}{ab^3}\left[(7-8k^2)E(k)-4(1-k^2)K(k)\right]. \quad (19)$$

$$\int_0^\infty K_1(ax)J_0(bx)\ x^5dx=\frac{3k^5}{ab^5}\left[(43-168k^2+128k^4)E(k)-4(1-k^2)\ (7-16k^2)K(k)\right]. \quad (20)$$

$$\int_0^\infty K_1(ax)J_0(bx)\ x^7dx=\frac{15k^7}{ab^7}\left[(337-2784k^2+5504k^4-3072k^6)E(k)\right.$$
$$\left.-8(1-k^2)\ (29-176k^2+192k^4)K(k)\right]. \quad (21)$$

$$\int_0^\infty K_1(ax)J_1(bx)\ dx=\frac{1}{ak}\left[E(k)-(1-k^2)K(k)\right]. \quad (22)$$

$$\int_0^\infty K_1(ax)J_1(bx)\ x^2dx=\frac{k}{ab^2}\left[(1-k^2)K(k)-(1-2k^2)\ \ E(k)\right]. \quad (23)$$

$$\int_0^\infty K_1(ax)J_1(bx)\ x^4dx=\frac{3k^3}{ab^4}\left[(1-8k^2)(1-k^2)K(k)-(1-16k^2+16k^4)E(k)\right]. \quad (24)$$

$$\int_0^\infty K_1(ax)J_1(bx)\ x^6dx=\frac{15k^5}{ab^6}\left[(1-k^2)(3-80k^2+128k^4)K(k)-(3-134k^2+384k^4-256k^6)E(k)\right]. \quad (25)$$

$$\int_0^\infty K_1(ax)J_2(bx)\ x\,dx=\frac{1}{abk}\left[(2-k^2)E(k)-2(1-k^2)K(k)\right]. \quad (26)$$

$$\int_0^\infty K_1(ax)J_2(bx)\ x^3dx=\frac{k}{ab^3}\left[2(1-k^2)(1+2k^2)K(k)-(2+3k^2-8k^4)E(k)\right]. \quad (27)$$

$$\int_0^\infty K_1(ax)J_2(bx)\ x^5dx=\frac{3k^3}{ab^5}\left[2(1-k^2)(1+6k^2-32k^4)K(k)-(2+11k^2-136k^4+128k^6)E(k)\right]. \quad (28)$$

$$\int_0^\infty K_1(ax)J_2(bx)\ x^{-1}dx=\frac{b}{3ak^3}\left[(2-k^2)E(k)-2(1-k^2)K(k)\right]. \quad (29)$$

$$\int_0^\infty K_1(ax)J_3(bx)\ dx=\frac{1}{3ak^3}\left[(8-7k^2)E(k)-(8-3k^2)(1-k^2)K(k)\right]. \quad (30)$$

$$\int_0^\infty K_1(ax)J_3(bx)\ x^2dx=\frac{1}{ab^2k}\left[(8-3k^2-2k^4)E(k)-(1-k^2)\ (8+k^2)K(k)\right]. \quad (31)$$

$$\int_0^\infty K_1(ax)J_3(bx)\ x^4dx=\frac{k}{ab^4}\left[(1-k^2)\ (8+13k^2+24k^4)K(k)-(8+9k^2+16k^4-48k^6)E(k)\right]. \quad (32)$$

$$\int_0^\infty K_1(ax)J_3(bx)\,x^{-2}dx=\frac{b^2}{45ak^5}\,[(8-3k^2-2k^4)E(k)-(1-k^2)\,(8+k^2)K(k)]. \quad (33)$$

$$\int_0^\infty K_2(ax)J_0(bx)\,x^2dx=\frac{k}{a^2b}\,[2(2-k^2)E(k)-(1-k^2)K(k)]. \quad (34)$$

$$\int_0^\infty K_2(ax)J_0(bx)\,x^4dx=\frac{k^3}{a^2b^3}\,[2(19-44k^2+24k^4)E(k)-(1-k^2)\,(23-24k^2)K(k)]. \quad (35)$$

$$\int_0^\infty K_2(ax)J_0(bx)\,x^6dx=\frac{3k^5}{a^2b^5}\,[2(158-923k^2+1408k^4-640k^6)$$
$$\times E(k)-(1-k^2)(211-848k^2+640k^4)K(k)]. \quad (36)$$

$$\int_0^\infty K_2(ax)J_1(bx)\,x\,dx=\frac{1}{a^2k}\,[(1+k^2)E(k)-(1-k^2)K(k)]. \quad (37)$$

$$\int_0^\infty K_2(ax)J_1(bx)\,x^3dx=\frac{k}{a^2b^2}\,[(1-k^2)(3-4k^2)K(k)-(3-13k^2+8k^4)E(k)]. \quad (38)$$

$$\int_0^\infty K_2(ax)J_1(bx)\,x^5dx=\frac{3k^3}{a^2b^4}\,[(1-k^2)(5-68k^2+64k^4)K(k)-(5-123k^2+248k^4-128k^6)E(k)]. \quad (39)$$

$$\int_0^\infty K_2(ax)J_2(bx)\,dx=\frac{b}{3a^2k^3}\,[(1-k^2)(2-3k^2)K(k)-2(1-2k^2)E(k)]. \quad (40)$$

$$\int_0^\infty K_2(ax)J_2(bx)\,x^2dx=\frac{1}{a^2bk}\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \quad (41)$$

$$\int_0^\infty K_2(ax)J_2(bx)\,x^4dx=\frac{3k}{a^2b^3}\,[(1-k^2)(2+5k^2-8k^4)K(k)-2(1-2k^2)(1+4k^2-4k^4)E(k)]. \quad (42)$$

$$\int_0^\infty K_2(ax)J_2(bx)\,x^6dx=\frac{15k^3}{a^2b^5}\,[(2+15k^2-144k^4+128k^6)\,(1-k^2)K(k)$$
$$-2(1+7k^2-135k^4+256k^6-128k^8)E(k)]. \quad (43)$$

$$\int_0^\infty K_2(ax)J_3(bx)\,x\,dx=\frac{1}{3a^2k^3}\,[(1-k^2)\,(8-9k^2)K(k)-(8-13k^2+3k^4)E(k)]. \quad (44)$$

$$\int_0^\infty K_2(ax)J_3(bx)\,x^3dx=\frac{1}{a^2b^2k}\,[(1+k^2)\,(8-13k^2+8k^4)E(k)-(1-k^2)\,(8-k^2-4k^4)K(k)]. \quad (45)$$

$$\int_0^\infty K_2(ax)J_3(bx)\,x^{-1}dx=\frac{b^2}{15a^2k^5}\,[(1-k^2)\,(8-9k^2)K(k)-(8-13k^2+3k^4)E(k)]. \quad (46)$$

$$\int_0^\infty K_3(ax)J_0(bx)\,x^3dx=\frac{k}{a^3b}\,[(23-23k^2+8k^4)E(k)-4(2-k^2)\,(1-k^2)K(k)]. \quad (47)$$

$$\int_0^\infty K_3(ax)J_0(bx)\,x^5dx=\frac{k^3}{a^3b^3}\,[(281-985k^2+1080k^4-384k^6)E(k)$$
$$-4(1-k^2)\,(44-93k^2+48k^4)K(k)]. \quad (48)$$

$$\int_0^\infty K_3(ax)J_1(bx)\ x^2dx=\frac{1}{a^3k}\left[(3+7k^2-2k^4)E(k)-(1-k^2)\ (3+k^2)K(k)\right]. \tag{49}$$

$$\int_0^\infty K_3(ax)J_1(bx)\ x^4dx=\frac{k}{a^3b^2}\left[(15-43k^2+24k^4)(1-k^2)K(k)\right.$$
$$\left.-(15-103k^2+128k^4-48k^6)E(k)\right]. \tag{50}$$

$$\int_0^\infty K_3(ax)J_2(bx)\ x\,dx=\frac{b}{3a^3k^3}\left[2(1-k^2)\ (1-3k^2)K(k)-(2-7k^2-3k^4)E(k)\right]. \tag{51}$$

$$\int_0^\infty K_3(ax)J_2(bx)\ x^3dx=\frac{1}{a^3bk}\left[(6-9k^2+19k^4-8k^6)E(k)-2(1-k^2)\ (3-3k^2+2k^4)K(k)\right]. \tag{52}$$

$$\int_0^\infty K_3(ax)J_3(bx)\ dx=\frac{b^2}{15a^3k^5}\left[(8-23k^2+23k^4)E(k)-(1-k^2)\ (8-19k^2+15k^4)K(k)\right]. \tag{53}$$

$$\int_0^\infty K_3(ax)J_3(bx)\ x^2dx=\frac{1}{3a^3k^3}\left[(8-15k^2+3k^4)\ (1-k^2)K(k)-(8-19k^2+9k^4-6k^6)E(k)\right]. \tag{54}$$

$$\int_0^\infty K_4(ax)J_0(bx)\ x^4dx=\frac{k}{a^4b}\left[8(11-11k^2+6k^4)\right.$$
$$\left.\times(2-k^2)E(k)-(1-k^2)\ (71-71k^2+24k^4)K(k)\right]. \tag{55}$$

$$\int_0^\infty K_4(ax)J_1(bx)\ x^3dx=\frac{1}{a^4k}\left[(15+58k^2-33k^4+8k^6)E(k)-(1-k^2)\ (15+13k^2-4k^4)K(k)\right]. \tag{56}$$

$$\int_0^\infty K_4(ax)J_2(bx)\ x^2dx=\frac{b}{a^4k^3}\left[(1-k^2)\ (2-9k^2-k^4)K(k)-2(1+k^2)\ (1-6k^2+k^4)E(k)\right]. \tag{57}$$

$$\int_0^\infty K_4(ax)J_3(bx)\ x\,dx=\frac{b^2}{15a^4k^5}\left[(8-33k^2+58k^4+15k^6)\right.$$
$$\left.\times E(k)-(1-k^2)\ (8-29k^2+45k^4)K(k)\right]. \tag{58}$$

6.4. $k^2=\left(\dfrac{b}{\sqrt{a^2+b^2}+a}\right)^2$

$$\int_0^\infty K_0(2ax)J_0^2(bx)\ dx=\frac{2k}{\pi b}\left[K(k)\right]^2. \qquad \text{OB 153 (2.37)} \tag{1}$$

$$\int_0^\infty K_0(2ax)J_1^2(bx)\ dx=\frac{2}{\pi bk}\left[K(k)-E(k)\right]^2. \tag{2}$$

$$\int_0^\infty K_0(2ax)J_2^2(bx)\ dx=\frac{2}{9\pi bk^3}\left[(2+k^2)K(k)-2(1+k^2)E(k)\right]^2. \tag{3}$$

$$\int_0^\infty K_0(2ax)J_3^2(bx)\ dx=\frac{2}{225\pi bk^5}\left[(8+3k^2+4k^4)K(k)-(8+7k^2+8k^4)E(k)\right]^2. \tag{4}$$

$$\int_0^\infty K_1(2ax)J_0^2(bx)\ x\,dx=\frac{1}{2\pi a\sqrt{a^2+b^2}}K(k)\left[2E(k)-(1-k^2)K(k)\right]. \tag{5}$$

$$\int_0^\infty K_1(2ax)J_1^2(bx)\ x\,dx=\frac{1}{2\pi k^2a\sqrt{a^2+b^2}}\left[K(k)-E(k)\right]\left[(1+k^2)E(k)-(1-k^2)K(k)\right]. \tag{6}$$

$$\int_0^\infty K_1(2ax)J_{\frac{3}{2}}^2(bx)\,x\,dx=\frac{1}{6\pi k^4 a\sqrt{a^2+b^2}}\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]$$
$$\times[(2+k^2)K(k)-2(1+k^2)E(k)]. \quad (7)$$

$$\int_0^\infty K_2(2ax)J_1^2(bx)\,dx=\frac{b}{6\pi a^2k^3}\,[(1+k^2)E(k)-(1-k^2)K(k)]^2. \quad (8)$$

$$\int_0^\infty K_2(2ax)J_{\frac{3}{2}}^2(bx)dx=\frac{b}{30\pi a^2k^5}\,[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]^2. \quad (9)$$

$$\int_0^\infty K_2(2ax)J_{\frac{5}{2}}^2(bx)dx=\frac{b}{630\pi a^2k^7}\,[(8-13k^2+8k^4)(1+k^2)E(k)$$
$$-(1-k^2)(8-k^2-4k^4)K(k)]^2. \quad (10)$$

$$\int_0^\infty K_3(2ax)J_1^2(bx)xdx=\frac{b^2}{24\pi k^4a^3\sqrt{a^2+b^2}}\,[(1+14k^2+k^4)E(k)$$
$$-(1-k^2)(1+7k^2)K(k)][(1+k^2)E(k)-(1-k^2)K(k)]. \quad (11)$$

6.5. $k^2=\left(\dfrac{b}{\sqrt{a^2+b^2}+a}\right)^2$

$$\int_0^\infty K_0^2(ax)J_0(2bx)dx=\frac{2k}{b}\,[K(k)]^2. \qquad \text{OB 23 (2.117)} \quad (1)$$

$$\int_0^\infty K_0^2(ax)J_1(2bx)xdx=\frac{1}{b\sqrt{a^2+b^2}}\,K(k)[K(k)-E(k)]. \quad (2)$$

$$\int_0^\infty K_0^2(ax)J_2(2bx)dx=\frac{2}{bk}\,[K(k)-E(k)]^2. \quad (3)$$

$$\int_0^\infty K_0^2(ax)J_3(2bx)xdx=\frac{1}{bk^2\sqrt{a^2+b^2}}\,[K(k)-E(k)][(2+k^2)K(k)-2(1+k^2)E(k)]. \quad (4)$$

$$\int_0^\infty K_0^2(ax)J_4(2bx)dx=\frac{2}{9bk^3}\,[(2+k^2)K(k)-2(1+k^2)E(k)]^2. \quad (5)$$

$$\int_0^\infty K_1^2(ax)J_1(2bx)xdx=\frac{b}{4k^2a^2\sqrt{a^2+b^2}}$$
$$\times[(1+k^2)E(k)-(1-k^2)K(k)][2E(k)-(1-k^2)K(k)]. \quad (6)$$

$$\int_0^\infty K_1^2(ax)J_2(2bx)dx=\frac{b}{6k^2a^2}\,[(1+k^2)E(k)-(1-k^2)K(k)]^2. \quad (7)$$

6.6 $k^2=\left(\frac{b}{\sqrt{a^2+b^2}+a}\right)^2$

$$\int_0^\infty K_0(2ax)J_1(bx)J_0(bx)xdx=\frac{\sqrt{a^2+b^2}}{\pi a^2 b}K(k)[K(k)-E(k)]. \tag{1}$$

$$\int_0^\infty K_0(2ax)J_2(bx)J_1(bx)xdx=\frac{\sqrt{a^2+b^2}}{\pi a^2 bk^2}[K(k)-E(k)][(2+k^2)K(k)-2(1+k^2)E(k)]. \tag{2}$$

$$\int_0^\infty K_0(2ax)J_3(bx)J_2(bx)xdx=\frac{\sqrt{a^2+b^2}}{9\pi a^2 bk^4}$$
$$\times[(2+k^2)K(k)-2(1+k^2)E(k)][(8+3k^2+4k^4)K(k)-(8+7k^2+8k^4)E(k)]. \tag{3}$$

$$\int_0^\infty K_2(2ax)J_1(bx)J_0(bx)xdx=\frac{b\sqrt{a^2+b^2}}{4\pi a^4k^2}[(1+k^2)E(k)-(1-k^2)K(k)][2E(k)-(1-k^2)K(k)]. \tag{4}$$

$$\int_0^\infty K_2(2ax)J_2(bx)J_1(bx)xdx=\frac{b\sqrt{a^2+b^2}}{12\pi a^4k^4}$$
$$\times[(1+k^2)E(k)-(1-k^2)K(k)][2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{5}$$

$$\int_0^\infty K_2(2ax)J_3(bx)J_2(bx)xdx=\frac{b\sqrt{a^2+b^2}}{60\pi a^4k^6}[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]$$
$$\times[(1+k^2)(8-13k^2+8k^4)E(k)-(1-k^2)(8-k^2-4k^4)K(k)]. \tag{6}$$

6.7. $k^2=\left(\frac{b}{\sqrt{a^2+b^2}+a}\right)^2$

$$\int_0^\infty K_1(ax)K_0(ax)J_0(2bx)xdx=\frac{1}{2a\sqrt{a^2+b^2}}K(k)[2E(k)-(1-k^2)K(k)]. \tag{1}$$

$$\int_0^\infty K_1(ax)K_0(ax)J_2(2bx)xdx=\frac{1}{2k^2a\sqrt{a^2+b^2}}[K(k)-E(k)][(1+k^2)E(k)-(1-k^2)K(k)]. \tag{2}$$

$$\int_0^\infty K_1(ax)K_0(ax)J_4(2bx)xdx=\frac{1}{6k^4a\sqrt{a^2+b^2}}[(2+k^2)K(k)-2(1+k^2)E(k)]$$
$$\times[2(1-k^2+k^4)E(k)-(1-k^2)(2-k^2)K(k)]. \tag{3}$$

$$\int_0^\infty K_2(ax)K_1(ax)J_2(2bx)xdx=\frac{b^2}{24k^4a^3\sqrt{a^2+b^2}}[(1+k^2)E(k)-(1-k^2)K(k)]$$
$$\times[(1+14k^2+k^4)E(k)-(1-k^2)(1+7k^2)K(k)]. \tag{4}$$

$$\int_0^\infty K_3(ax)K_2(ax)J_4(2bx)xdx=\frac{b^4}{3360k^8a^5\sqrt{a^2+b^2}}[(1-k^2)(2-9k^2-k^4)K(k)$$
$$-2(1+k^2)(1-6k^2+k^4)E(k)][(1-k^2)(2-21k^2-108k^4-k^6)K(k)$$
$$-2(1-11k^2-108k^4-11k^6+k^8)E(k)]. \tag{5}$$

6.8. $k^2=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2$

$$\int_0^\infty K_0(ax)K_0(bx)J_0(ax)J_0(bx)xdx=\frac{1}{2(a^2+b^2)}K(k). \tag{1}$$

The author wishes to express his hearty gratitude to the late Dr. Minoru Nakagami for his invaluable advice and encouragement. His thanks are also due to Mr. Yukihiko Miyawaki and Mr. Masayuki Sawada who gave him generous assistance in carrying out this work.

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For preparation of this paper the author referred to general expressions of these types of integrals in terms of hypergeometric functions or Legendre functions. The section numbers and the general expressions referred there are:

2.1 WA 385 (2)(3),ET II 29 (6); 2.2 deduced from ET I 196 (12); 2.3,2.4 LU 319 (28); 2.5,2.6 WA 389 (1)(2), LU 319 (26)(27); 2.7 WA 401 (2),407 (1); 2.8 ET II 52 (33); 3.1 WA 385 (4),ET II 105 (2); 3.3 ET I 332 (37); 4.1 GW 199 (6a),OL 149 (15.8); 5.1 ET II 131 (23); 5.2,5.3 ET II 145 (49); 6.1 deduced from ET I 198 (30); 6.2,6.3 WA 410 (1); 6.4 ET I 138 (18)(19); 6.5 ET II 66 (27)(28); 6.6 ET II 138 (20); 6.7 ET II 67 (29); 6.8 ET II 373 (10).

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(Paper 78B3–408)

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The Convex Hull of the Transposition Matrices

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The convex hull of the n by n transposition matrices is characterized as the set of symmetric doubly stochastic matrices with trace $n-2$. A similar characterization (with trace $\geq n-2$) is given for the convex hull of the union of transposition matrices and the identity matrix.

Key words: Combinatorial analysis; convex set; linear inequalitites; permutations.

Many combinatorial optimization problems can be expressed as requiring the extremization of a linear function over some finite set S of points in a real N-dimensional space. To bring the theoretical and computational resources of linear programming to bear, it is necessary to characterize the convex hull $K(S)$ of S as the solution-set of a "nicely structured" family of linear inequalities and equations.

The outstanding example, arising in connection with the assignment problem of operations research, has $S=S_n$, the set of n by n permutation matrices (regarded as points in n^2 dimensional space). Here a well-known theorem[1] (Birkhoff-Hoffman-von Neumann-Wielandt et al.) identifies $K(S_n)$ as the set of all n by n doubly stochastic matrices $X=(x_{ij})$, i.e. matrices with nonnegative entries and with each row and column summing to 1. It is expected that a similar characterization of $K(C_n)$, where C_n is the set of all cyclic permutation matrices, would be valuable in connection with the traveling saleman problem, but no such characterization has been given as yet.

For a given n let $\Pi_c (1 \leq c \leq n)$ denote the set of n by n permutation matrices for which the decomposition of the associated permutation into disjoint cycles contains exactly c cycles (including cycles of length one). Since $C_n=\Pi_1$, the remark ending the last paragraph suggests looking at the "other end" of the sequence $\{\Pi_c\}_{c=1}^{n}$. The situations for Π_n and Π_{n-1} are simple, and form the subject of this note. Clearly Π_n consists of the identity matrix I_n, so that $K(\Pi_n)=\{I_n\}$. We go on to characterize $K(\Pi_{n-1})$ as well as $K(\Pi_{n-1} \cup \Pi_n)=K(\Pi_{n-1} \cup \{I_n\})$. Note that Π_{n-1} consists of the $n(n-1)/2$ transposition matrices $T_{pq} (1 \leq p < q \leq n)$ defined by

$$(T_{pq})_{ij}=1 \qquad \text{if } (i,j)=(p,q) \text{ or } (q,p),$$

$$(T_{pq})_{ii}=1 \qquad \text{for } i \neq p, q,$$

$$(T_{pq})_{ij}=0 \qquad \text{otherwise.}$$

THEOREM 1. $K(\Pi_{n-1})$ *consists of all symmetric doubly stochastic matrices with trace* $n-2$.

THEOREM 2. $K(\Pi_{n-1} \cup \{I_n\})$ *consists of all symmetric doubly stochastic matrices with trace* $\geq n-2$.

To begin the proof, note that each member of Π_{n-1} (of $\Pi_{n-1} \cup \{I_n\}$) is a symmetric doubly stochastic matrix with trace $n-2$ (with trace $\geq n-2$). It readily follows that the same is true for each member of $K(\Pi_{n-1})$ (of $K(\Pi_{n-1} \cup \{I_n\})$).

AMS Subject Classification: 05B20, 15A51.

[1] Cf. Chap. 5: H. J. Ryser, Combinatorial Mathematics, Carus Mathematical Monograph No. 14, MAA, John Wiley & Sons 1963.

Next, let $X = (x_{ij})$ be any symmetric doubly stochastic matrix. Note that

$$\sum_{p}\sum_{q>p} X_{pq}=\frac{1}{2}\sum_{p,\,q\neq p} X_{pq}=\frac{1}{2}[n-\mathrm{tr}\,(X)]. \tag{1}$$

The matrix

$$Y=X-\sum_{p}\sum_{q>p} x_{pq}T_{pq}$$

is readily seen to be diagonal, with (i, i) entry

$$\begin{aligned} y_{ii} &= x_{ii}-\Sigma\{x_{pq}: p\neq i,\ q\neq i,\ q>p\} \\ &= x_{ii}-\sum_{p<i}\Big(\sum_{q>p}(x_{pq}-x_{pi})\Big)-\sum_{p>i}\sum_{q>p} x_{pq} \\ &= x_{ii}+\sum_{p<i} x_{pi}-\sum_{p\neq i}\sum_{q>p} x_{pq} \\ &= \sum_{p<i} x_{pi}-\Big[\sum_{p}\sum_{q>p} X_{pq}-\sum_{q>i} X_{iq}\Big] \\ &= \Big[\sum_{p\neq i} x_{pi}+\sum_{q>i} x_{qi}\Big]-\sum_{p}\sum_{q>p} x_{pq}. \end{aligned}$$

Since column i of X sums to 1, the last result together with (1) yields

$$y_{ii}=1-\frac{1}{2}[n-\mathrm{tr}(X)]=\frac{1}{2}[\mathrm{tr}(X)-(n-2)]=\delta, \tag{2}$$

where the final equation defines δ.

If $\mathrm{tr}(X)=n-2$, then (2) yields $Y=0$ and (1) shows that $X=\Sigma x_{pq}T_{pq}$ lies in $K(\Pi_{n-1})$, completing the proof of Theorem 1. If $\mathrm{tr}(X)\geq n-2$, then $\delta\geq 0$; since (2) shows that $Y=\delta I_n$ and since (1) and (2) yield $\Sigma x_{pq}+\delta=1$, it follows that $X=\Sigma x_{pq}T_{pq}+\delta I_n$ lies in $K(\Pi_{n-1}\cup\{I_n\})$, completing the proof of Theorem 2.

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Computational Experience With an Algorithm for Finding the k Shortest Paths in a Network*

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A particular computer implementation of the Double-Sweep method for calculating the k shortest paths in a network is described. Results are presented for a series of computational experiments performed on rectangular grid networks.

Key words: Double-Sweep method; graph; k shortest paths; network; network algorithms; shortest path.

1. Introduction

A common task which arises in analyzing a system of interconnected elements or *network* is that of calculating *shortest paths*—i.e., routes through the system whose total length or cost is as small as possible. Such calculations occur quite naturally in the context of transportation and communication networks. In applications such as these, it is sometimes desirable to have knowledge of the *k shortest paths* in contrast to simply a shortest path. For example, the knowledge of good alternative routes (as opposed to just the shortest one) can be used by transportation planners to model more realistically the flow of vehicular traffic on a road network. Or, as a second example, the routing of messages through a communications network when some routes are temporarily obstructed can be based on the best alternative routes which are available.

Several algorithms have been traditionally employed in order to determine the k shortest paths between specified nodes of a network (such paths may in fact contain repeated nodes). An excellent survey of these algorithms is provided by the review article of Dreyfus [1][1]. More recently, several new methods for performing such calculations have been proposed [2, 4]. These methods are based on a fairly strong analogy which exists between the solution of network path problems and traditional techniques for solving ordinary linear equations. On the basis of preliminary theoretical and computational evidence, one of these (the Double-Sweep method) emerged [4] as a reasonably effective procedure for calculating k shortest paths between a given node and all other nodes in a network.[2] The purpose of this report is to describe a particular implementation of the Double-Sweep method in FORTRAN V[3] and to present a body of computational results for a practically important class of networks (namely, those with a rectangular grid topology).

2. The Double-Sweep Method

Suppose that $G = (\mathcal{N}, \mathcal{A})$ is a finite directed *network* in which the real number l_{ij} denotes the length of *arc* $(i, j) \in \mathcal{A}$ joining *nodes* $i, j \in \mathcal{N}$. Node i of the arc (i, j) is said to be *incident to*

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The algorithms presented in [2] are appropriate for calculating k shortest paths between all pairs of nodes; because of storage limitations, the application of these algorithms is limited to networks having at most a few hundred nodes.

[3] FORTRAN V is UNIVAC's augmented version of standard FORTRAN IV.

node j, while node j is said to be *incident from* node i. A *path* from node i to node j is an ordered sequence of arcs $[(i, i_1), (i_1, i_2), \ldots, (i_{m-1}, j)]$ in the network; a path is termed *elementary* if all nodes appearing along the path are distinct. A *circuit* is a path whose *starting node* i and *ending node* j coincide. The *length* of a path is defined to be the arithmetic sum of the arc lengths l_{ij} along the path.

The problem under investigation here is that of determining, among *all* paths extending between two specified nodes, those paths having the smallest, the second smallest, . . . and the kth smallest length. It is emphasized that these k *shortest paths* (that is, paths whose lengths are shortest, second shortest, . . . or kth shortest) are not required to be elementary. The method to be discussed here will allow the calculation of the k shortest paths from a given source node to all other nodes in the network. It is assumed that

(1) All circuits in the network have positive length, and
(2) The network contains no self-loops: that is, circuits of the form $[(i, i)]$.

The *Double-Sweep* method calculates the k shortest path lengths from a particular source node to all n nodes of the given network by means of alternating forward and backward passes. A precise statement of the method, together with a proof of its validity, can be found in [4]. Basically, the method begins with an initial guess as to the k shortest path lengths and successively improves the current guess to obtain an even better guess. During the forward pass, the current path lengths to each node $j = 1, \ldots, n$ are modified by using those nodes $i < j$ which are incident to node j. If the sum of a current path length to some node i and the arc length l_{ij} provides a shorter path length to node j than any which is currently known, then the corresponding path lengths to node j are updated to give an improved estimate of the k shortest path lengths. A similar procedure is followed during the backward pass, except that now only nodes $i > j$ are considered with $j = n, \ldots, 1$.

The alternating forward and backward passes are continued until no improvement in the path lengths to any node can be made. Convergence in this sense will always be achieved in a finite number of steps—in fact, in at most $nk+1$ forward and $nk+1$ backward passes. When convergence does obtain, the k shortest path lengths to each node j in the network will have been found. From such path length information, the actual paths corresponding to any of the k shortest path lengths can be determined by a backward path tracing procedure.

In essence, this path tracing procedure is based on the following fact. Namely, if a tth shortest path π of length l from node i to node j passes through node r, then the subpath of π extending from node i to node r is a qth shortest path for some q, $1 \leq q \leq t$. This fact can be used to determine the penultimate node r on a tth shortest path of known length l from node i to node j. Indeed, any such node r can be found by forming the quantity $l - l_{rj}$ for all nodes r incident to node j and determining if this quantity appears as a qth shortest path length ($q \leq t$) for node r. If so, then there is a tth shortest path of length l whose final arc is (r, j); otherwise, no such path exists. This idea is repeatedly used, in the manner of a backtrack program, to produce all paths from i to j with the length l, and ultimately all the k shortest paths from node i to node j.

It should be pointed out that, while the Double-Sweep method will work perfectly well if circuits of zero length are present, the preceding path tracing procedure may encounter some difficulties if such zero length circuits exist. The essential difficulty is that there can then be an infinite number of paths having the same length, so the generation of *all* of these paths is clearly impossible. Accordingly, the simple backtracking system described above could possibly cycle indefinitely unless certain precautions are taken. For the sake of retaining simplicity, then, the (quite reasonable) assumption has been made that all circuits have strictly positive length.

3. Program Implementation

The calculation of k shortest paths from a particular source node can be accomplished through the use of the four subprograms listed in section 5. The subroutine INPUT allows the description

of the given network to be entered, the subroutines DSWP and XMULT are used to calculate the k shortest path lengths, and the subroutine TRACE enables the actual tracing out of the k shortest paths. Certain details of the specific implementation used will be discussed in this section.

The first issue concerns the choice of the starting guess with which to initiate the Double-Sweep method, as several choices are possible. It has proved convenient to assign, as the initial approximation, k "infinite" path lengths to each node (the value used for infinity was $INF = 99999999$), save for the source node NS which receives the k-vector of path lengths $(0, INF, \ldots, INF)$. At any step of the process, the k-vector associated with each node will contain the k shortest path lengths found so far from node NS to that node. Moreover, these k path lengths are always distinct (apart from infinite values) and are always arranged in strictly increasing order. Such an ordering allows the following two computationally important observations to be made.

(1) If the value INF is encountered in some component of a k-vector, then all subsequent components of the k-vector also contain INF values. Therefore, when updating the k-vector for node j during a forward or backward pass, the k-vector for a node i incident to j need only be scanned as far as the first occurrence of an INF value since an infinite value cannot possibly yield an improved path length for node j.

(2) If IXV, the sum of some current path length in the k-vector for node i and the arc length l_{ij}, is greater than or equal to the maximum element of the k-vector for node j, then no improvement in the latter k-vector by use of the former can possibly be made. Therefore, it is appropriate to keep track of the current maximum element MAX of the k-vector for node j. If IXV is less than MAX then it is possible for an improvement to be made, as long as the value IXV does not already occur in the k-vector for node j (only *distinct* path lengths are retained).

The use of these two observations allows for a substantial reduction in the amount of computational effort required to update the current path lengths, as compared to the use of some general sorting routine to find the k smallest elements in a list. Since such updating comprises the major computational requirement of the Double-Sweep method, it has proved advantageous to keep the components of each k-vector in strictly increasing order. These updating steps, corresponding to appropriate "matrix multiplications" as defined in [4], are performed by the subroutine XMULT, called as required by the subroutine DSWP.

Together, the subprograms DSWP and XMULT enable the calculation of the K shortest path lengths from any given source node NS. In some applications, these path lengths may be all the information that is required by the user. In others, the actual paths joining various pairs of nodes are also needed. To accomplish this latter task, the subroutine TRACE is used to produce all paths from node NS to node NF having any of the K shortest path lengths from NS to NF. As presently implemented, paths containing up to 5000 arcs will be generated; an error message will indicate when this condition is not fulfilled.

The subroutine INPUT allows a description of the given network to be read in from external unit number 5 together with values for relevant parameters. The network description is achieved by specifying for each arc of the network a record containing its starting node, its ending node and its length. The records are assumed to be sorted in increasing order by arc ending node; moreover, the nodes are assumed to be numbered consecutively from 1 to N. The totality of such network description records is followed by a blank record and then a sequence of parameter specification records. Each block of parameter specification records consists of a path calculation record followed by any number, possibly zero, of path tracing records. The path calculation record gives values for K, NS and $IMAX$. The parameter $IMAX$ is the maximum number of Double-Sweep iterations [4] allowed before a mandatory termination of DSWP is imposed; for most cases, a value of $IMAX = 100$ should suffice. Each path tracing record gives values for NF and $PMAX$, which allow the user to trace out actual paths from node NS to node NF if required; at most $PMAX$ such paths will be determined. Several (but at least one) parameter specification blocks can be accommodated so that various values for K, NS and NF can be explored in the given network. The final record of the input stream is required to be blank.

[4] Each forward or backward pass constitutes an iteration.

The form of input acceptable to the program can be concisely specified by the following *BNF* formula system [3].

⟨input⟩ ::= ⟨network part⟩ ⟨blank record⟩ ⟨parameter part⟩ ⟨blank record⟩
⟨network part⟩ ::= ⟨arc record⟩|⟨network part⟩ ⟨arc record⟩
⟨parameter part⟩ ::= ⟨parameter block⟩|⟨parameter part⟩ ⟨parameter block⟩
⟨parameter block⟩ ::= ⟨path calcn record⟩|⟨parameter block⟩ ⟨path trace record⟩

Definitions for the various records mentioned above are provided below; the columns of a record which are not explicitly mentioned are assumed to be blank.

⟨blank record⟩:
⟨arc record⟩: Cols. 1–10 = Arc Starting Node
Cols. 11–20 = Arc Ending Node
Cols. 21–30 = Arc Length
⟨path calcn record⟩: Cols. 1–5 = K
Cols. 6–10 = NS
Cols. 11–15 = $IMAX$
⟨path trace record⟩: Cols. 1–5 = NF
Cols. 6–10 = $PMAX$

The values given to K, NS, $IMAX$, NF, and $PMAX$ are all specified by the user. In addition, the following ranges are assumed for the indicated variables.

N = number of nodes in the network: $2 \leq N \leq 1000$.
MU = number of arcs (I, J) with $I < J$: $0 \leq MU \leq 5000$.
ML = number of arcs (I, J) with $I > J$: $0 \leq ML \leq 5000$.
K = number of distinct path lengths required:[5] $2 \leq K \leq 20$.

Thus, at most $K = 20$ shortest path lengths from a given node in a network with up to 1000 nodes and 10,000 arcs can be handled by the currently programmed version of the algorithm. The approximate storage requirements for the four subroutines comprising the package are given in table 1; the storage requirements appropriate for arbitrary values of N, K, and $M = MU + ML$ are given in table 2. If there is insufficient storage available on a particular computer system to retain all the subroutines in core simultaneously, it is possible to break up the execution sequence by performing first the path length calculation (DSWP/XMULT) and next the path tracing calculation (TRACE). This is a feasible strategy because essentially two different network representations are used in these two distinct calculation phases.

TABLE 1. *Approximate number of storage locations required by various subprograms*

Subprogram	Storage for common blocks used				Program instruction storage	Total required storage[a]
	BLK1	BLK2	BLK3	BLK4		
INPUT	22003	2		21001	268	43274
DSWP	22003	2	20000		267	42272
XMULT		2	20000		214	20216
TRACE		2	20000	21001	15310	56313
ALL	22003	2	20000	21001	16059	79065

[a] Exclusive of system routines (requiring approximately 5700 locations for the UNIVAC 1108, EXEC 8 operating system used).

[5] Only minor program changes are necessary to allow the case $K = 1$.

TABLE 2. *Approximate number of storage locations required by common blocks and subprograms in terms of* N, K *and* M = MU + ML

Storage for common blocks		Total storage for subprograms[a]	
BLK1	$2N+2M+3$	INPUT	$3N+4M+274$
BLK2	2	DSWP	$NK+2N+2M+272$
BLK3	NK	XMULT	$NK+216$
BLK4	$N+2M+1$	TRACE	$NK+N+2M+15313$
		ALL	$NK+3N+4M+16065$

[a] Exclusive of system routines (requiring approximately 5700 locations for the UNIVAC 1108, EXEC 8 operating system used).

TABLE 3. *Listing of sample card input*

		1	65
		1	38
		2	78
		2	82
		2	78
		3	79
		3	96
		3	55
		4	87
		4	12
		5	24
		5	9
		5	18
		6	45
		6	48
		6	22
		6	23
		7	8
		7	41
		7	44
		7	58
		8	29
		8	47
		8	51
		9	74
		9	88
		10	69
		10	40
		10	48
		11	32
		11	35
		11	93
		12	14
		12	30
(blank)			
5	12	30	
1	20		
(blank)			

TABLE 4. *Listing of sample printed output*

$K=5$ shortest path lengths from node 12

To node					
1	164	205	211	220	221
2	195	232	236	241	242
3	150	159	191	200	206
4	63	104	128	145	154
5	126	167	173	182	183
6	117	158	164	173	174
7	95	136	151	158	160
8	51	92	116	133	142
9	200	229	241	247	256
10	141	175	186	206	216
11	93	127	158	168	176
12	0	65	106	123	130

Number of iterations required for convergence = 6

The K shortest paths from node 12 to node 1

Path	Length	Node sequence							
1	164	12	8	7	6	5	1		
2	205	12	8	4	8	7	6	5	1
3	211	12	11	10	6	5	1		
4	220	12	11	7	6	5	1		
5	221	12	8	7	6	5	6	5	1

The output produced from execution of the subroutine DSWP takes the form of a tabular listing of the K shortest path lengths from node NS to all nodes in the network. In addition, the number of Double-Sweep iterations required for convergence of the algorithm is printed. Each time the subroutine TRACE is called, the required K shortest paths (together with their respective lengths) are printed out as the appropriate sequences of nodes leading from NS to NF. All printing is assumed to be done through external unit number 6.

Documented listings of the four subprograms INPUT, DSWP, XMULT and TRACE are given in section 5. Further details about this implementation of the Double-Sweep method, with a path tracing facility, can be obtained by referring to these listings.

Finally, we describe some sample input for the k shortest path package as well as the resulting output. The particular network chosen for illustrative purposes is shown in figure 1. For this net-

work of 12 nodes and 34 arcs, it is required to find the five shortest path lengths from node 12 to all nodes and the actual paths between node 12 and node 1. Accordingly, $K = 5$, $NS = 12$, and $NF = 1$; the values chosen for the other two input parameters are $IMAX = 30$, $PMAX = 20$. Table 3 displays the form of card input required for this particular problem. Note that the network description cards are listed in order of increasing arc ending node number. Table 4 displays the resulting output from the computer program.

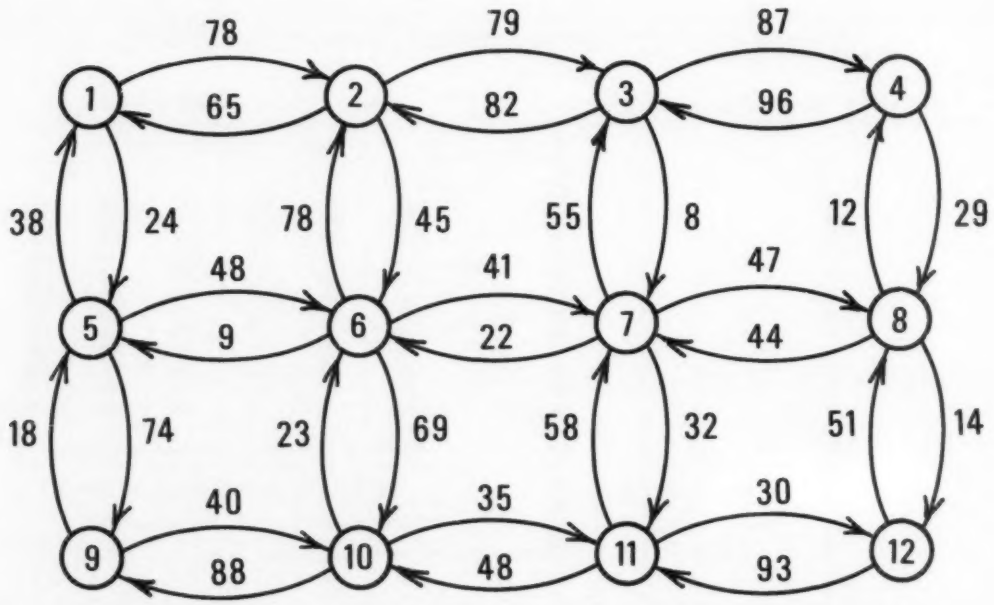

FIGURE 1. *An illustrative network.*

4. Computational Results

The Double-Sweep method has been previously shown [4] to possess certain theoretical and computational advantages over other algorithms for finding the k shortest paths from a given node. The current implementation of this method, embodied in the four subroutines discussed earlier, has undergone testing for a variety of different sample networks. This section will present in particular the results of a number of computational experiments performed on rectangular grid networks. Such networks were chosen for detailed investigation since grid-like networks arise quite frequently in representing, for example, a city street system.

A general $P \times Q$ grid network consists of $N = PQ$ nodes (which are assumed to be numbered consecutively from left to right and top to bottom) and $4PQ - 2(P + Q)$ arcs. The integral lengths of each arc are generated from a uniform distribution over the interval 1 to *RANGE*. (See fig. 1 for an illustrative 3×4 grid network.) The currently implemented version of the Double-Sweep method was used to obtain the K shortest path lengths from node NS to all nodes of various grid networks; no tracing of paths was considered in these computational experiments. The response characteristics of elapsed computation time on a UNIVAC 1108 were studied as different parameters of the problem were varied.

First, a series of four 24×24 grid networks $G_1 - G_4$ were created by drawing four different random samples of arc lengths from the uniform distribution on 1 to 100. Figures 2–4 show how for three distinct source nodes the variation of computation time with K is affected by the particular arc length sample G_i chosen. Since the arc length sample chosen does have a noticeable effect on computation time, it was decided that subsequently all results would be averaged over four arc length samples. Figure 5 shows how the *average* computation time varies, quite nearly quadratically, with K for three distinct source nodes. In fact, table 5 displays the second-degree polynomials which give the best (least squares) fit to the curves of figure 5 over the range $2 \leq K \leq 20$. From the values obtained for the residual standard deviations, these second-degree polynomials produce an excellent fit to the observations; essentially, then, the average computation time varies quadratically with K. Such behavior is not unexpected since the doubling of K, for instance, not only increases the computation per iteration (potentially twice as many sums IXV must now be formed for each node) but also tends to increase the number of iterations required for convergence (the equality of $2k$-vectors automatically ensures the equality of their first k components).

TABLE 5. *Best fit second-degree polynomials to figure 5 curves*

NS	Second-degree fitted polynomial	Residual standard deviation[a](s)
1	$0.8457 + 0.1616\,K + 0.0260\,K^2$	0.1369
300	$1.6111 + 0.0761\,K + 0.0425\,K^2$	0.2151
576	$0.7747 + 0.1603\,K + 0.0249\,K^2$	0.1121

[a] Residual standard deviation $= \sqrt{\frac{\Sigma (y_i - \hat{y}_i)^2}{n-3}}$.

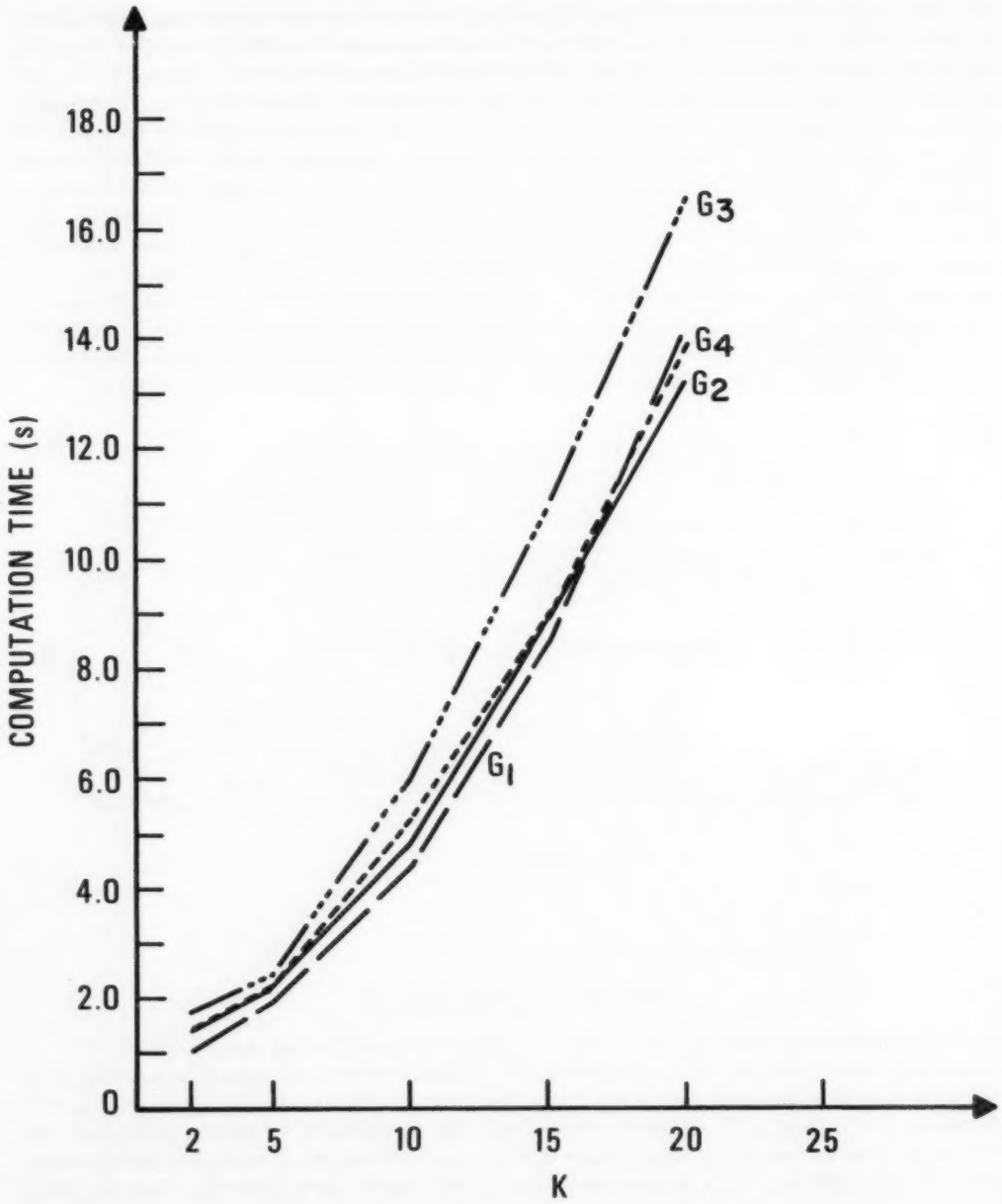

FIGURE 2. *Computation times for four 24 × 24 grids ("corner" source node 1,* RANGE = *100).*

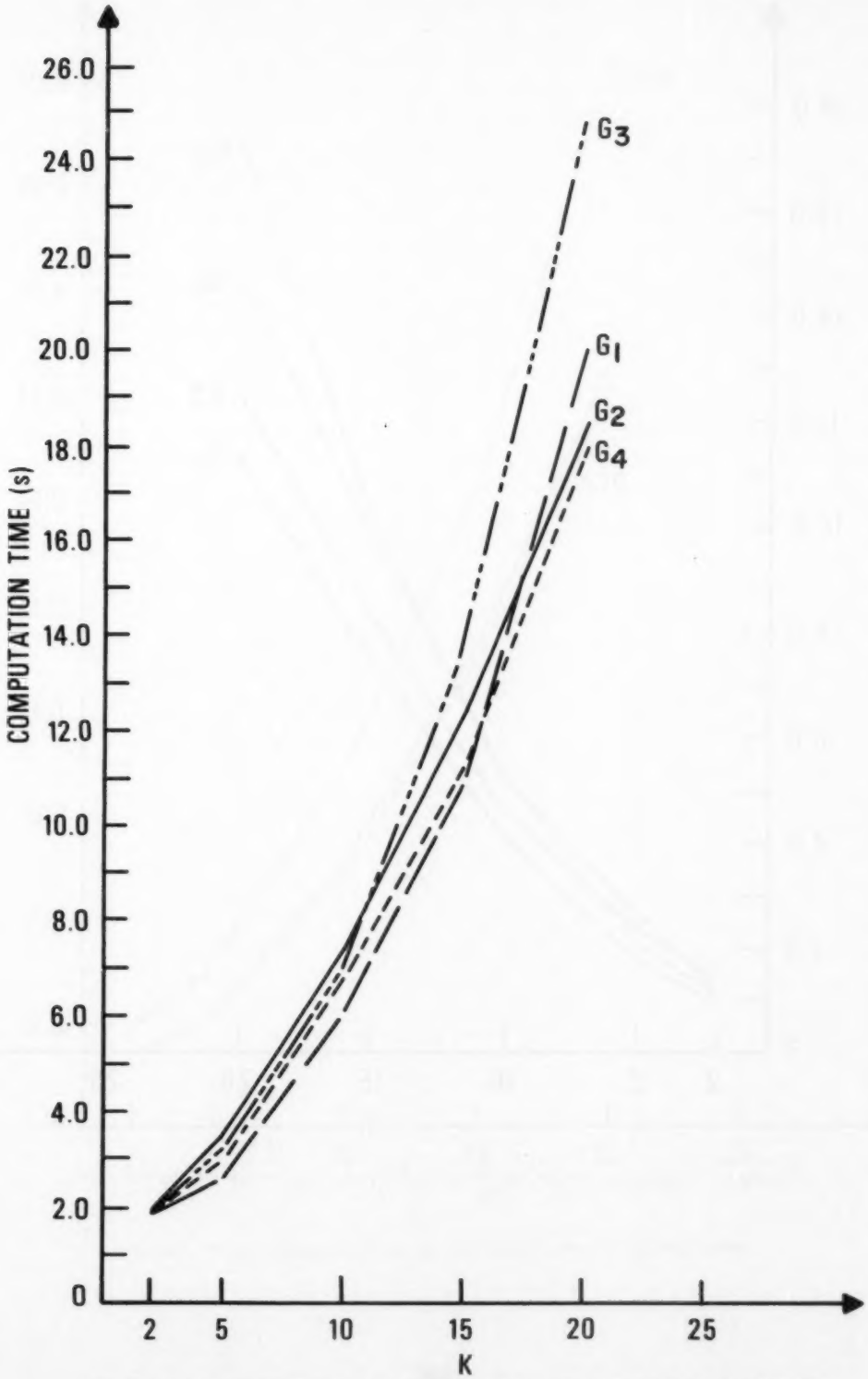

FIGURE 3. *Computation times for four 24 × 24 grids ("central" source node 300,* RANGE = *100).*

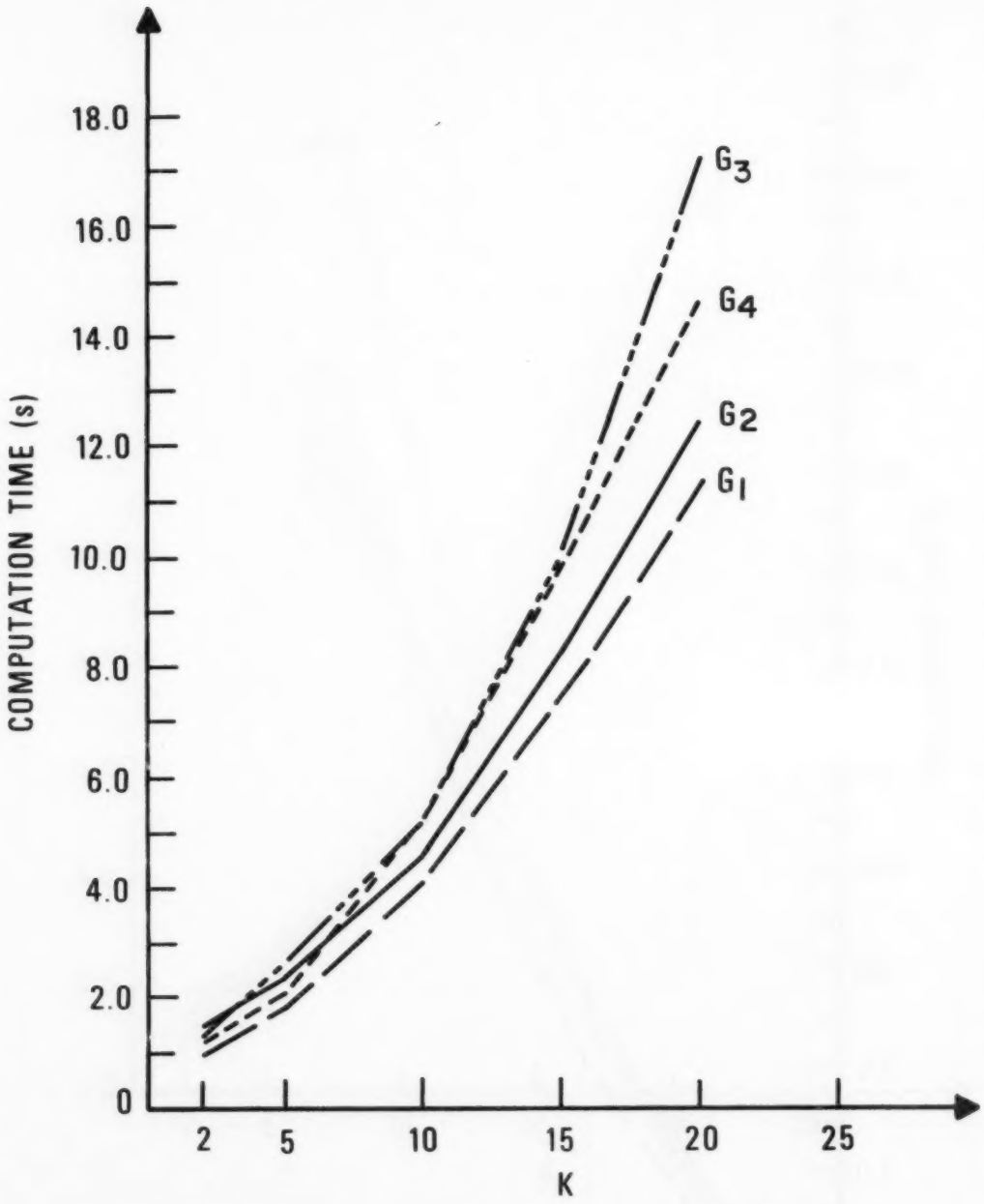

FIGURE 4. *Computation times for four 24 × 24 grids ("corner" source node 576,* RANGE = *100).*

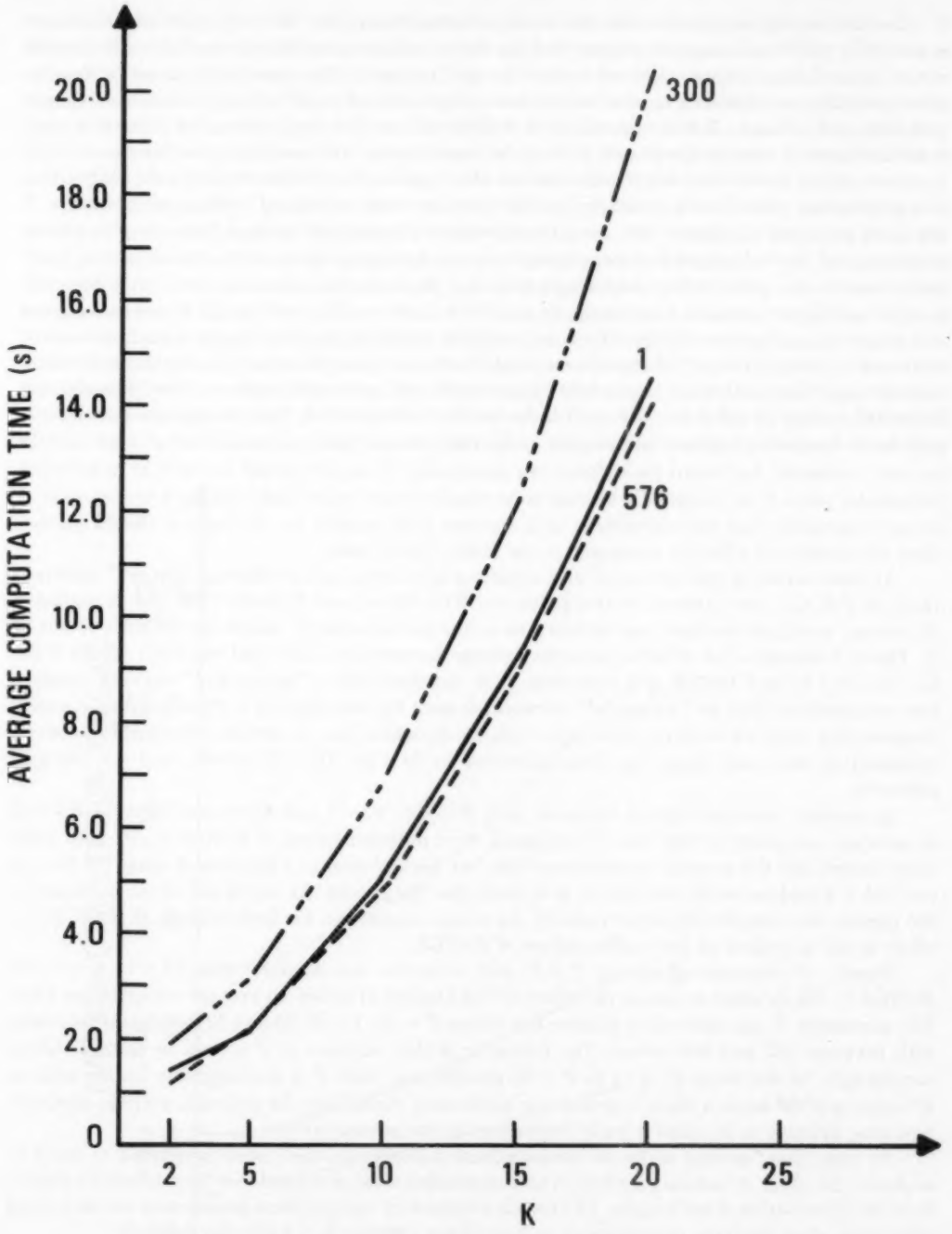

FIGURE 5. *Average computation time for source nodes 1, 300, 576* (RANGE = *100*).

In addition, figure 5 shows that the average computation time for a centrally located source node ($NS = 300$) is consistently higher than for the boundary nodes $NS = 1$ and $NS = 576$ (located at the upper left and lower right corners of the grid network). One possible explanation for this observed difference between central and corner nodes is based on the concept of a *nonretrograde* path in a grid network: that is, a path whose tracing only involves two orthogonal directions (e.g., north and east). A nonretrograde path from node 1 only involves the south and east directions, while a nonretrograde path from node N only involves the north and west directions. On the other hand, a nonretrograde path from a centrally located node can use any pair of orthogonal directions. It has been observed in practice that for grid networks with randomly generated arc lengths a large percentage of the calculated k shortest paths are nonretrograde paths, while the remaining such paths tend to be quite nearly nonretrograde paths. Because the nodes are numbered from left to right and top to bottom, a nonretrograde path from node 1 will have its length correctly assigned in a single forward pass of the Double-Sweep method. Similarly, a nonretrograde path from node N will have its length correctly assigned in a single backward pass. However, a nonretrograde north/east or south/west path from a centrally located node will generally require several forward and backward passes in order for its length to be correctly determined. For example, in a north/east path each horizontal segment will require a separate forward pass and each vertical segment will require a separate backward pass. Since the processing of nonretrograde (as well as nearly nonretrograde) paths from a centrally located node requires more effort than that for a corner node, it seems reasonable that the calculation of k shortest path lengths for the former should involve more computational effort as compared to the latter type of node.

Another series of grid networks all containing 576 nodes but of differing "shapes" (different ratios of P to Q) were created for test purposes. With $NS = 1$ and $RANGE = 100$, the dependence of average computation time was explored as a function of network shape for different values of K. Figure 6 displays the relation between average computation time and $\log(P/Q)$ for 24×24, 32×18, 64×9, and 144×4 grid networks. It is apparent that a "square-like" network requires less computation than an "elongated" network, despite the fact that for a given number of nodes a square-like network contains more arcs than an elongated one. A similar relationship between computation time and shape has been observed in 24×24, 18×32, 9×64, and 4×144 grid networks.

In another sequence of test networks with $N = 576$, $NS = 1$ and $K = 5$, the effect of *RANGE* on average computation time was investigated. Four different values of *RANGE* (1, 10, 100, 1000) were chosen and the average computation time has been plotted as a function of range for 24×24 and 144×4 grid networks (see fig. 7). It is clear that the greater the variability of the arc lengths, the greater the computation time required. As seems reasonable, for larger values of *RANGE*, the effect is not as radical as for smaller values of *RANGE*.

Finally, a sequence of square $P \times P$ grid networks was studied with $NS = 1$, $K = 5$ and $RANGE = 100$ in order to assess the effect of the number of nodes on average computation time. The parameter P was allowed to assume the values $P = 10$, 15, 20, 24 and 30 (producing networks with between 100 and 900 nodes). The influence of this variation in P is seen in figure 8. Quite surprisingly, in the range $P = 15$ to $P = 30$ the increase with P is demonstrably linear, with an R^2 value of 0.988 when a linear regression is performed. Recall that, by contrast, average computation time appears to be quadratically dependent on the parameter K.

In sum, then, several series of computational experiments have been performed in order to evaluate the effect of various parameters on computation time, averaged over four different samples from the distribution of arc lengths. The results obtained by varying these parameters can be helpful guidelines when planning computations to investigate changes to a particular network.

In all of the above experiments, the numbering of nodes was taken to be the usual left-to-right, top-to-bottom numbering described earlier. Since each pass of the Double-Sweep method processes the nodes in the fixed order $1, \ldots, N$ (for a forward pass) or the fixed order $N, \ldots, 1$ (for a backward pass), the actual numbering of nodes used can potentially affect the convergence characteristics of this method. Accordingly, a 20×20 grid network with specific randomly generated arc

lengths in the range 1 to 100 was investigated for various node numbering schemes. To begin, 10 different randomly generated permutations were used to renumber the nodes of this 20×20 grid network. The $K=5$ shortest path lengths were calculated from the source node located at the upper left corner of each of the 10 randomly renumbered networks, yielding a mean computation time of 2.488 s with a standard deviation of 0.220 s. By contrast, when the usual left-to-right, top-to-bottom numbering scheme was used, the same $K=5$ shortest path lengths were produced in only 1.213 s, approximately half the average time for randomly numbered networks.

Furthermore, when the nodes were renumbered consecutively from left to right along odd-numbered rows and right to left along even-numbered rows, the required computing time was 1.683 s. On the other hand, when the nodes were renumbered consecutively from top to bottom along odd-numbered columns and bottom to top along even-numbered columns, the required computing time was 1.504 s. The main conclusion to be drawn from such times is that the ordering of nodes can have a pronounced effect on the efficiency of the Double-Sweep method. In particular, a systematic numbering scheme is to be preferred to a random numbering of the nodes. This result accords well with our intuition, since in a systematic numbering scheme the effect of changing the k-vector associated with a given node will be passed on rather quickly to some neighboring nodes; there is certainly no guarantee that such will be the case for randomly assigned numbering schemes.

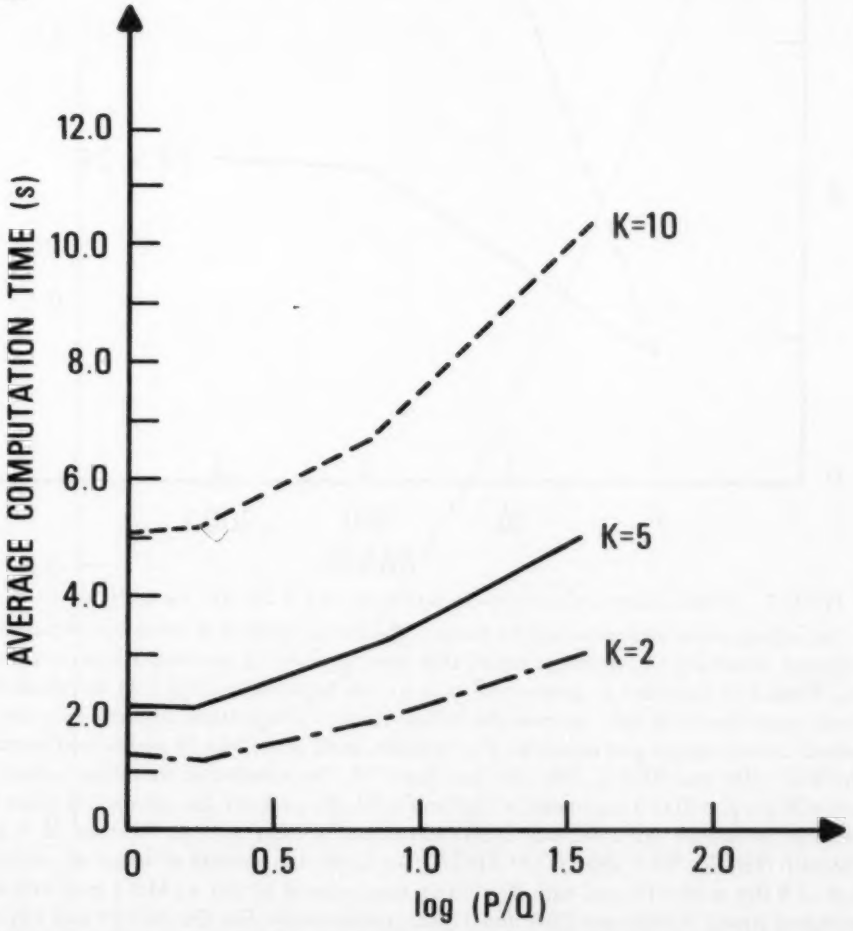

FIGURE 6. *Average computation time for 24×24, 32×18, 64×9 and 144×4 grids* (NS = *1*, RANGE = *100*, K = *2, 5, 10*).

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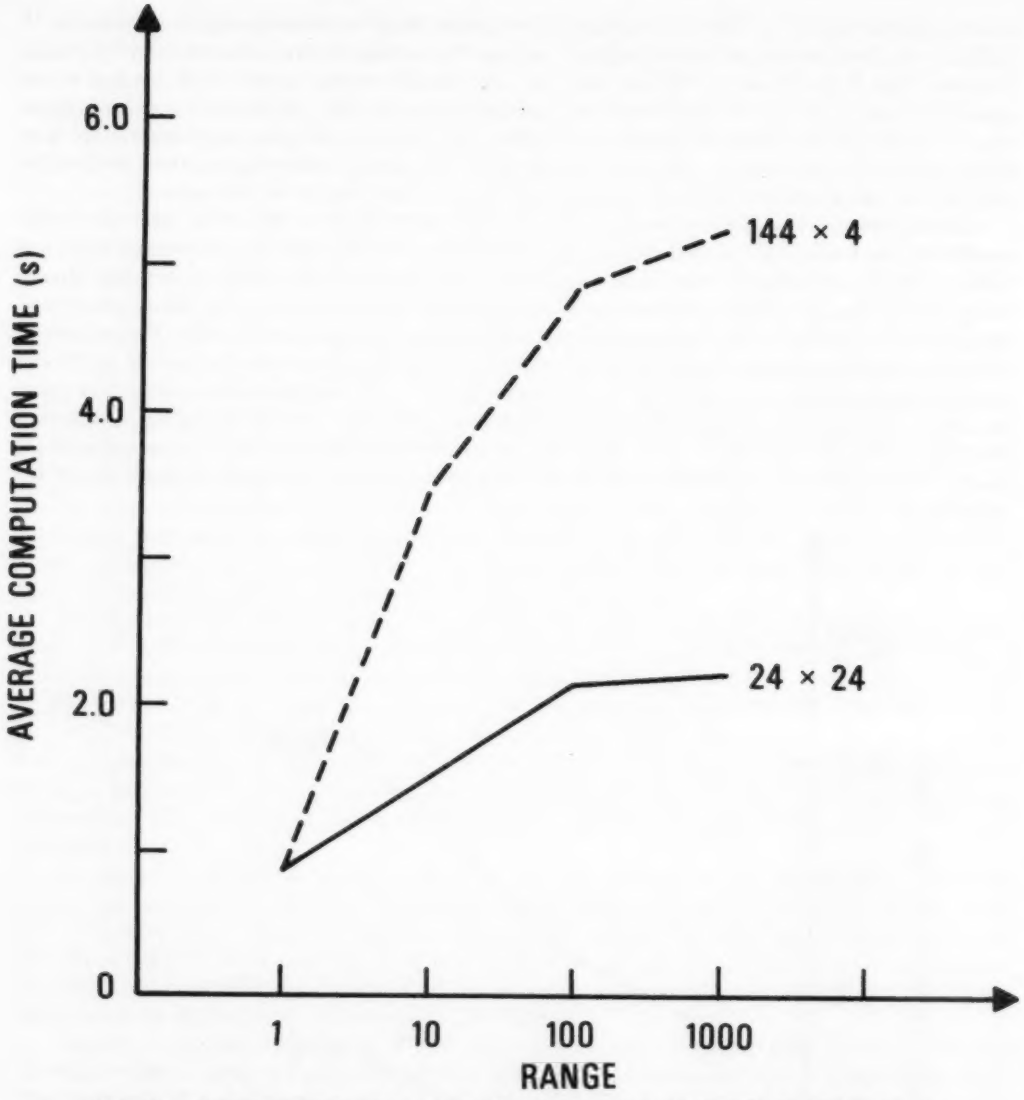

FIGURE 7. *Average computation time related to* RANGE *for 24×24 and 144×4 grids* (NS = *1*, K = *5*).

In concluding, some indication will be given of the actual number of iterations required for the Double-Sweep algorithm to converge; recall that each forward or backward pass constitutes an iteration. While the algorithm is guaranteed to converge in at most $2(NK+1)$ iterations [4], this theoretical upper bound usually exceeds the actual number of iterations required by a few orders of magnitude in our sample grid networks. For example, in all of the 24×24 grid networks generated with $RANGE = 100$ and $NS = 1$, 300, 576 (see figs.2–4), the number of iterations varied from a minimum of 9 (for $K = 2$) to a maximum of 25 (for $K = 20$). By contrast, the theoretical upper bounds for these two situations are 2306 and 23,042 iterations, respectively. In the case of $P \times Q$ grid networks with $N = 576$, $NS = 1$ and $RANGE = 100$ (see fig. 6), the number of iterations varied from a minimum of 9 (for a 32×18 grid with $K = 2$) to a maximum of 37 (for a 144×4 grid with $K = 10$); the theoretical upper bounds are 2306 and 11,522, respectively. For the 24×24 and 144×4 grid networks with $NS = 1$ and $K = 5$ (see fig. 7), the number of iterations varied from a minimum of

5 (for all 24×24 and 144×4 grids with $RANGE = 1$) to a maximum of 39 iterations (for a 144×4 grid with $RANGE = 1000$); the theoretical upper bound for this situation is 5762 iterations. Finally, even for the 900 node square grids with $NS = 1$, $K = 5$ and $RANGE = 100$ (see fig. 8), the number of iterations was at most 19, compared to the theoretical upper bound of 9002. In all these cases, then, the number of iterations required for convergence is really quite modest; in addition, the theoretical upper bounds are seen to give very little qualitative or quantitative information about the actual number of iterations needed.

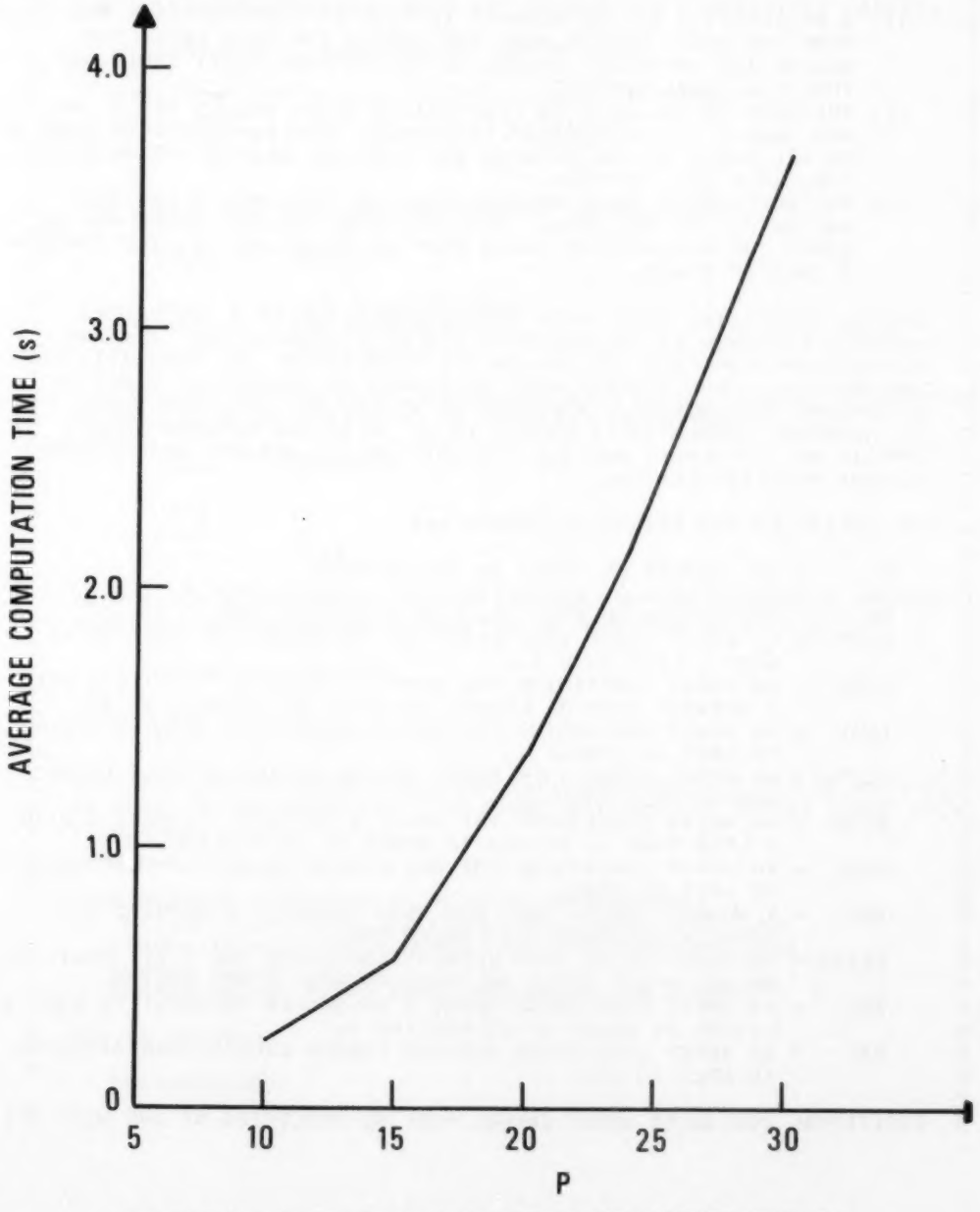

FIGURE 8. *Average computation time for five* P × P *grids* (NS = *1*, RANGE = *100*, K = *5*).

```
      SUBROUTINE INPUT
C
C
C  THIS SUBROUTINE INVOLVES THE FOLLOWING INPUT SEQUENCE.
C
C    (1) A DESCRIPTION OF THE NETWORK IS READ IN FROM UNIT 5 - THE
C        STARTING NODE, ENDING NODE AND LENGTH FOR EACH ARC.  THE
C        ARRAYS AND VARIABLES NEEDED BY SUBROUTINES DSWP, XMULT AND
C        TRACE ARE THEN CREATED.
C    (2) THE NEXT RECORD READ IN FROM UNIT 5 GIVES VALUES FOR K, NS
C        AND IMAX.  THE K SHORTEST (DISTINCT) PATH LENGTHS FROM NODE NS
C        TO ALL NODES OF THE NETWORK ARE THEN CALCULATED AND PRINTED
C        THROUGH A CALL OF DSWP.
C    (3) THE SUCCEEDING INPUT RECORDS READ IN FROM UNIT 5 INDICATE
C        VALUES FOR NF AND PMAX.  FOR EACH SUCH RECORD (THERE MAY BE
C        NONE) THE APPROPRIATE PATHS FROM NS TO NF ARE PRINTED THROUGH
C        A CALL OF TRACE.
C
C  SEVERAL SUCCESSIVE INPUT DATA SETS, CONSISTING OF A (2) RECORD
C  POSSIBLY FOLLOWED BY (3) RECORDS, CAN BE ACCOMMODATED.  A BLANK
C  RECORD SHOULD PRECEDE AND FOLLOW THE TOTALITY OF ALL SUCH (2), (3)
C  COMBINATIONS.  THE NETWORK MUST BE SORTED IN INCREASING ORDER BY
C  ARC ENDING NODE NUMBER.  MOREOVER IT IS ASSUMED THAT THE NODES
C  ARE NUMBERED SEQUENTIALLY FROM 1 TO N.  ALSO THE NETWORK SHOULD
C  CONTAIN NO SELF-LOOPS AND ALL CIRCUITS IN THE NETWORK ARE REQUIRED
C  TO HAVE POSITIVE LENGTH.
C
C  THE VARIABLES AND ARRAYS IN COMMON ARE
C
C     N      = THE NUMBER OF NODES IN THE NETWORK.
C     MU     = THE NUMBER OF ARCS (I,J) WITH I LESS THAN J.
C     ML     = THE NUMBER OF ARCS (I,J) WITH J LESS THAN I.
C     LLEN   = AN ARRAY WHOSE J-TH ENTRY IS THE NUMBER OF ARCS (I,J)
C              WITH J LESS THAN I.
C     LINC   = AN ARRAY CONTAINING THE NODES I INCIDENT TO NODE J WITH
C              I GREATER THAN J, LISTED IN ORDER OF INCREASING J.
C     LVAL   = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C              TO ARCS IN LINC.
C     ULEN   = AN ARRAY WHOSE J-TH ENTRY IS THE NUMBER OF ARCS (I,J)
C              WITH I LESS THAN J.
C     UINC   = AN ARRAY CONTAINING THE NODES I INCIDENT TO NODE J WITH
C              I LESS THAN J, LISTED IN ORDER OF INCREASING J.
C     UVAL   = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C              TO ARCS IN UINC.
C     INF    = A NUMBER LARGER THAN ANY PATH LENGTH.  A NONEXISTENT
C              PATH IS ASSIGNED THE LENGTH INF.
C     START  = AN ARRAY WHOSE J-TH ELEMENT INDICATES THE FIRST POSITION
C              ON INC WHERE NODES INCIDENT TO NODE J ARE LISTED.
C     INC    = AN ARRAY CONTAINING NODES I WHICH ARE INCIDENT TO NODE J,
C              LISTED IN ORDER OF INCREASING J.
C     VAL    = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C              TO ARCS IN INC.
C
C  ADDITIONAL VARIABLES WHOSE VALUES MUST BE SPECIFIED BY THE USER ARE
```

```
C
C      K     = THE NUMBER OF DISTINCT PATH LENGTHS REQUIRED.
C      IMAX  = THE MAXIMUM NUMBER OF DOUBLE-SWEEP ITERATIONS ALLOWED.
C              FOR MOST CASES IMAX = 100 SHOULD SUFFICE.
C      NS,NF = THE INITIAL AND FINAL NODES OF ALL K SHORTEST PATHS TO
C              BE GENERATED.
C      PMAX  = THE MAXIMUM NUMBER OF PATHS TO BE GENERATED BETWEEN
C              NODES NS AND NF.
C
C   THE FOLLOWING VARIABLES MUST LIE WITHIN THE (INCLUSIVE) RANGES.
C
C         VARIABLE   LOWER BOUND   UPPER BOUND
C
C             N           2           1000
C            MU           0           5000
C            ML           0           5000
C             K           2             20
C
C   THE FORMAT FOR THE INPUT NETWORK IS AS FOLLOWS.
C
C         COLS.      CONTENTS
C
C          1-10      ARC STARTING NODE
C         11-20      ARC ENDING NODE
C         21-30      ARC LENGTH
C
C   THE NETWORK DESCRIPTION IS FOLLOWED BY A BLANK RECORD.  SUCCEEDING
C   RECORDS GIVE VALUES FOR K, NS AND IMAX ACCORDING TO
C
C         COLS.      CONTENTS
C
C          1-5       K
C          6-10      NS
C         11-15      IMAX
C
C   EACH (K,NS,IMAX) RECORD CAN BE FOLLOWED BY ANY NUMBER OF RECORDS
C   (POSSIBLY NONE) GIVING VALUES FOR NF AND PMAX ACCORDING TO
C
C         COLS.      CONTENTS
C
C          1-5       NF
C          6-10      PMAX
C
C   THIS PATTERN OF (K,NS,IMAX) AND (NF,PMAX) RECORDS CAN BE REPEATED
C   AS OFTEN AS IS REQUIRED.  THE FINAL DATA RECORD OF THE INPUT DECK
C   IS BLANK.
C
C
      DIMENSION LLEN(1000),LINC(5000),LVAL(5000)
      INTEGER ULEN(1000),UINC(5000),UVAL(5000),START,VAL
      COMMON /BLK1/ N,MU,ML,LLEN,LINC,LVAL,ULEN,UINC,UVA
      COMMON /BLK2/ INF,K
      COMMON /BLK4/ START(1001),INC(10000),VAL(10000)
C
C   INF IS DEFINED.
C
      INF=99999999
C
```

```
C  AS THE INPUT NETWORK IS READ IN, THE VARIABLES AND ARRAYS NEEDED
C  BY DSWP AND TRACE ARE CREATED.
C
      J=0
      MU=0
      ML=0
      NPREV=0
      N=0
    1 READ (5,800) NB,NA,LEN
  800 FORMAT(3I10)
      IF(NA.GT.N) N=NA
      IF(NB.GT.N) N=NB
      IF(NA.EQ.NPREV) GO TO 10
      IF(NA.EQ.NPREV+1) GO TO 3
      IF(NA.EQ.0) GO TO 30
      L1=NPREV+1
      L2=NA-1
      DO 2 L=L1,L2
      START(L)=0
      ULEN(L)=0
    2 LLEN(L)=0
    3 IF(J.EQ.0) GO TO 5
      ULEN(NPREV)=JU
      LLEN(NPREV)=JL
    5 START(NA)=J+1
      JU=0
      JL=0
      NPREV=NA
   10 J=J+1
      INC(J)=NB
      VAL(J)=LEN
      IF(NB.GT.NA) GO TO 20
      MU=MU+1
      UINC(MU)=NB
      UVAL(MU)=LEN
      JU=JU+1
      GO TO 1
   20 ML=ML+1
      LINC(ML)=NB
      LVAL(ML)=LEN
      JL=JL+1
      GO TO 1
   30 START(NPREV+1)=J+1
      ULEN(NPREV)=JU
      LLEN(NPREV)=JL
C
C  THE (K,NS,IMAX) AND (NF,PMAX) DATA RECORDS ARE SUCCESSIVELY READ.
C
   40 READ (5,801) I1,I2,I3
  801 FORMAT(3I5)
      IF(I1.EQ.0) GO TO 100
      IF(I3.EQ.0) GO TO 50
      K=I1
      NS=I2
C
C  THE K SHORTEST DISTINCT PATH LENGTHS FROM NODE NS TO ALL NODES OF
C  THE NETWORK ARE CALCULATED.
C
```

```
      CALL DSWP(NS,I3)
      GO TO 40
C
C  UP TO PMAX OF THE PATHS HAVING THE K SHORTEST PATH LENGTHS FROM NODE
C  NS TO NODE NF ARE DETERMINED.
C
   50 CALL TRACE(NS,I1,I2)
      GO TO 40
  100 RETURN
      END
```

```
      SUBROUTINE DSWP(NS,IMAX)
C
C
C  THIS SUBROUTINE IMPLEMENTS THE DOUBLE-SWEEP METHOD IN ORDER TO
C  CALCULATE THE K SHORTEST (DISTINCT) PATH LENGTHS FROM NODE NS TO
C  ALL NODES OF A NETWORK.  THE NETWORK IS ASSUMED TO CONTAIN NO
C  SELF-LOOPS AND ALL CIRCUITS IN THE NETWORK ARE REQUIRED TO HAVE
C  POSITIVE LENGTH.  THE REQUIRED K SHORTEST PATH INFORMATION IS
C  PRINTED OUT ON UNIT 6.
C
C  THE VARIABLES IN THE CALLING SEQUENCE ARE
C
C     NS   = THE NODE FROM WHICH K SHORTEST PATH LENGTHS TO ALL NODES
C            ARE REQUIRED.
C     IMAX = THE MAXIMUM NUMBER OF DOUBLE-SWEEP ITERATIONS ALLOWED.
C
C  THE VARIABLES AND ARRAYS IN COMMON ARE
C
C     N    = THE NUMBER OF NODES IN THE NETWORK. (NODES ARE ASSUMED
C            NUMBERED SEQUENTIALLY FROM 1 TO N.)
C     MU   = THE NUMBER OF ARCS (I,J) WITH I LESS THAN J.
C     ML   = THE NUMBER OF ARCS (I,J) WITH J LESS THAN I.
C     LLEN = AN ARRAY WHOSE J-TH ENTRY IS THE NUMBER OF ARCS (I,J)
C            WITH J LESS THAN I.
C     LINC = AN ARRAY CONTAINING THE NODES I INCIDENT TO NODE J WITH
C            I GREATER THAN J, LISTED IN ORDER OF INCREASING J.
C     LVAL = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C            TO ARCS IN LINC.
C     ULEN = AN ARRAY WHOSE J-TH ENTRY IS THE NUMBER OF ARCS (I,J)
C            WITH I LESS THAN J.
C     UINC = AN ARRAY CONTAINING THE NODES I INCIDENT TO NODE J WITH
C            I LESS THAN J, LISTED IN ORDER OF INCREASING J.
C     UVAL = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C            TO ARCS IN UINC.
C     INF  = A NUMBER LARGER THAN ANY PATH LENGTH.  A NONEXISTENT
C            PATH IS ASSIGNED THE LENGTH INF.
C     K    = THE NUMBER OF DISTINCT PATH LENGTHS REQUIRED.
C     X    = AN ARRAY WHOSE (I,J)-TH ENTRY WILL EVENTUALLY CONTAIN
C            THE J-TH SHORTEST (DISTINCT) PATH LENGTH FROM NODE NS
C            TO NODE I.
C
C  ADDITIONAL VARIABLES ARE
C
C     ITNS = THE NUMBER OF ITERATIONS PERFORMED.
C     INDX = A VARIABLE RETURNED FROM XMULT WHICH = 1 IF CONVERGENCE
C            HAS OBTAINED AND = 0 OTHERWISE.
C
C  THE FOLLOWING VARIABLES MUST LIE WITHIN THE (INCLUSIVE) RANGES
C
C        VARIABLE   LOWER BOUND   UPPER BOUND
C
C           N            2           1000
C          MU            0           5000
C          ML            0           5000
```

```
C            K           2            20
C
C
      DIMENSION LLEN(1000),LINC(5000),LVAL(5000)
      INTEGER ULEN(1000),UINC(5000),UVAL(5000),X
      COMMON /BLK1/ N,MU,ML,LLEN,LINC,LVAL,ULEN,UINC,UVAL
      COMMON /BLK2/ INF,K
      COMMON /BLK3/ X(1000,20)
      N1=N-1
C
C  THE INITIAL APPROXIMATION MATRIX X IS FORMED.
C
      DO 20 I=1,N
      DO 20 J=1,K
   20 X(I,J)=INF
      X(NS,1)=0
      ITNS=1
C
C  THE CURRENT X IS MODIFIED THROUGH MATRIX MULTIPLICATION WITH THE
C  LOWER TRIANGULAR PORTION OF THE ARC LENGTH MATRIX.
C
   30 IFIN=ML
      INDX=1
      DO 40 I=N1,1,-1
      IF(LLEN(I).EQ.0) GO TO 40
      IS=IFIN-LLEN(I)+1
      CALL XMULT(I,IS,IFIN,LINC,LVAL,INDX)
      IFIN=IS-1
   40 CONTINUE
      IF(ITNS.EQ.1) GO TO 50
C
C  TEST FOR CONVERGENCE.
C
      IF(INDX.EQ.1) GO TO 100
C
C  THE CURRENT X IS MODIFIED THROUGH MATRIX MULTIPLICATION WITH THE
C  UPPER TRIANGULAR PORTION OF THE ARC LENGTH MATRIX.
C
   50 ITNS=ITNS+1
      IS=1
      INDX=1
      DO 60 I=2,N
      IF(ULEN(I).EQ.0) GO TO 60
      IFIN=IS+ULEN(I)-1
      CALL XMULT(I,IS,IFIN,UINC,UVAL,INDX)
      IS=IFIN+1
   60 CONTINUE
C
C  TEST FOR CONVERGENCE.
C
      IF(INDX.EQ.1) GO TO 100
      ITNS=ITNS+1
C
C  A TEST IS MADE TO SEE IF TOO MANY ITERATIONS HAVE BEEN PERFORMED.
C
      IF(ITNS.LT.IMAX) GO TO 30
      WRITE (6,900) IMAX
  900 FORMAT(' NUMBER OF ITERATIONS EXCEEDS',I5)
```

```
      GO TO 200
C
C  THE SOLUTION MATRIX X IS PRINTED OUT ON UNIT 6, TOGETHER WITH THE
C  VALUES FOR K, NS AND ITNS.
C
  100 WRITE (6,901) K,NS
  901 FORMAT(1H1,10X,'K=',I3,' SHORTEST PATH LENGTHS FROM NODE',I4//2X,
     1'TO'/1X,'NODE'//)
      DO 130 I=1,N
  130 WRITE (6,902) I,(X(I,J),J=1,K)
  902 FORMAT(' ',I3,6X,(10I9))
      WRITE (6,903) ITNS
  903 FORMAT(//1HO,'NUMBER OF ITERATIONS REQUIRED FOR CONVERGENCE =',I5)
  200 RETURN
      END
```

```
      SUBROUTINE XMULT(I,IS,IFIN,INC,VAL,INDX)
C
C
C  THIS SUBROUTINE WILL PERFORM THE APPROPRIATE MATRIX MULTIPLICATION
C  OF X BY THE I-TH COLUMN OF THE LOWER (OR UPPER) PORTION OF THE ARC
C  LENGTH MATRIX.  IN EFFECT, THE CURRENT K SHORTEST PATH LENGTHS
C  ASSOCIATED WITH NODE I ARE ADJUSTED BY CONSIDERING NODES WHICH
C  ARE INCIDENT TO NODE I.  IF AN IMPROVEMENT CAN BE MADE, THE VARIABLE
C  INDX WILL RETURN WITH THE VALUE 0.
C
C  THE VARIABLES AND ARRAYS IN THE CALLING SEQUENCE ARE
C
C     I    = THE COLUMN OF THE ARC LENGTH MATRIX TO BE MULTIPLIED ON
C            THE LEFT BY X.  THE CURRENT K SHORTEST PATH LENGTHS TO
C            NODE I WILL BE IMPROVED IF POSSIBLE.
C     IS   = THE STARTING POSITION IN LISTS INC AND VAL WHERE ARC
C            INFORMATION FOR NODE I CAN BE FOUND.
C     IFIN = THE FINAL POSITION IN LISTS INC AND VAL WHERE ARC
C            INFORMATION FOR NODE I CAN BE FOUND.
C     INC  = AN ARRAY CONTAINING NODE INCIDENCE INFORMATION, EITHER
C            LINC OR UINC.
C     VAL  = AN ARRAY CONTAINING ARC LENGTH INFORMATION, EITHER
C            LVAL OR UVAL.
C     INDX = A VARIABLE WHICH IS SET EQUAL TO 0 IF AN IMPROVEMENT CAN
C            BE MADE IN THE K SHORTEST PATH LENGTHS TO NODE I.
C
C  THE VARIABLES AND ARRAYS IN COMMON ARE
C
C     INF  = A NUMBER LARGER THAN ANY PATH LENGTH.  A NONEXISTENT PATH
C            IS ASSIGNED THE LENGTH INF.
C     K    = THE NUMBER OF DISTINCT PATH LENGTHS REQUIRED.
C     X    = THE CURRENT APPROXIMATION MATRIX FOR K SHORTEST PATH
C            LENGTHS FROM A GIVEN NODE TO ALL NODES OF THE NETWORK.
C
C  ADDITIONAL VARIABLES AND ARRAYS ARE
C
C     A    = AN AUXILIARY ARRAY USED IN FINDING THE K SMALLEST DISTINCT
C            ELEMENTS OF A SET.
C     MAX  = THE CURRENT MAXIMUM ELEMENT OF ARRAY A.
C     IXV  = A FEASIBLE PATH LENGTH TO NODE I FROM THE GIVEN NODE.
C
C
      DIMENSION INC(5000)
      INTEGER VAL(5000),A(20),X
      COMMON /BLK2/ INF,K
      COMMON /BLK3/ X(1000,20)
C
C  INITIALIZE A TO THE CURRENT K SHORTEST PATH LENGTHS FOR NODE I, IN
C  STRICTLY INCREASING ORDER.
C
      DO 10 J=1,K
   10 A(J)=X(I,J)
      MAX=A(K)
C
```

```
C  EACH NODE OF INC INCIDENT TO NODE I IS EXAMINED.
C
      DO 100 L=IS,IFIN
      II=INC(L)
      IV=VAL(L)
C
C  TEST TO SEE WHETHER IXV IS TOO LARGE TO BE INSERTED INTO A.
C
      DO 90 M=1,K
      IX=X(II,M)
      IF(IX.GE.INF) GO TO 100
      IXV=IX+IV
      IF(IXV.GE.MAX) GO TO 100
C
C  IDENTIFY THE POSITION INTO WHICH IXV CAN BE INSERTED.
C
      DO 30 J=K,2,-1
      IF(IXV-A(J-1)) 30,90,50
   30 CONTINUE
      J=1
   50 JJ=K
   70 IF(JJ.LE.J) GO TO 80
      A(JJ)=A(JJ-1)
      JJ=JJ-1
      GO TO 70
   80 A(J)=IXV
C
C  IF AN INSERTION HAS BEEN MADE IN A, SET INDX = 0.
C
      INDX=0
      MAX=A(K)
   90 CONTINUE
  100 CONTINUE
      IF(INDX.EQ.1) GO TO 120
C
C  UPDATE THE K SHORTEST PATH LENGTHS TO NODE I.
C
      DO 110 J=1,K
  110 X(I,J)=A(J)
  120 RETURN
      END
```

```
      SUBROUTINE TRACE(NS,NF,PMAX)
C
C
C  THIS SUBROUTINE WILL TRACE OUT THE PATHS CORRESPONDING TO THE K
C  DISTINCT SHORTEST PATH LENGTHS FROM NODE NS TO NODE NF.  AT MOST
C  PMAX SUCH PATHS WILL BE GENERATED.  IT IS ASSUMED THAT ALL CIRCUITS
C  IN THE NETWORK HAVE POSITIVE LENGTH.  MOREOVER ONLY PATHS HAVING
C  AT MOST 5000 ARCS WILL BE PRODUCED.  AN ERROR MESSAGE WILL INDICATE
C  WHEN THIS CONDITION IS NOT FULFILLED.  THE REQUIRED PATHS BETWEEN
C  NODES NS AND NF ARE PRINTED OUT ON UNIT 6.
C
C  THE VARIABLES IN THE CALLING SEQUENCE ARE
C
C     NS,NF = THE INITIAL AND FINAL NODES OF ALL K SHORTEST PATHS
C             BEING GENERATED.
C     PMAX  = THE MAXIMUM NUMBER OF PATHS TO BE GENERATED BETWEEN
C             NODES NS AND NF.
C
C  THE VARIABLES AND ARRAYS IN COMMON ARE
C
C     INF   = A NUMBER LARGER THAN ANY PATH LENGTH.  A NONEXISTENT PATH
C             IS ASSIGNED THE LENGTH INF.
C     K     = THE NUMBER OF DISTINCT PATH LENGTHS REQUIRED.  K SHOULD
C             LIE IN THE RANGE 2 TO 20 INCLUSIVE.
C     X     = AN ARRAY WHOSE (I,J)-TH ENTRY IS THE J-TH SHORTEST
C             (DISTINCT) PATH LENGTH FROM NODE NS TO NODE I.
C     START = AN ARRAY WHOSE J-TH ELEMENT INDICATES THE FIRST POSITION
C             ON INC WHERE NODES INCIDENT TO NODE J ARE LISTED.
C     INC   = AN ARRAY CONTAINING NODES I WHICH ARE INCIDENT TO NODE J,
C             LISTED IN ORDER OF INCREASING J.
C     VAL   = AN ARRAY CONTAINING THE ARC LENGTH VALUES CORRESPONDING
C             TO ARCS IN INC.
C
C  ADDITIONAL VARIABLES AND ARRAYS ARE
C
C     JJ    = INDEX OF THE PATH LENGTH FROM NS TO NF BEING EXPLORED.
C             JJ CAN TAKE ON VALUES FROM 1 TO K.
C     NP    = THE NUMBER OF PATHS FROM NS TO NF FOUND.
C     KK    = CURRENT POSITION OF LIST P.
C     P     = AN ARRAY CONTAINING NODES ON A POSSIBLE PATH FROM NS TO NF.
C     Q     = AN ARRAY WHOSE I-TH ELEMENT GIVES THE POSITION, RELATIVE
C             TO START, OF NODE P(I) ON THE INC LIST FOR P(I-1).
C     PV    = AN ARRAY WHOSE I-TH ELEMENT IS THE ARC LENGTH EXTENDING
C             FROM NODE P(I) TO NODE P(I-1).
C
C
      INTEGER P(5000),Q(5000),PV(5000),START,VAL,X,PMAX
      COMMON /BLK2/ INF,K
      COMMON /BLK3/ X(1000,20)
      COMMON /BLK4/ START(1001),INC(10000),VAL(10000)
C
C  INITIALIZATION PHASE.
C
      DO 10 I=1,5000
```

```
      P(I)=0
      Q(I)=0
   10 PV(I)=0
      JJ=1
      IF(NS.EQ.NF) JJ=2
      NP=0
      IF(X(NF,JJ).LT.INF) GO TO 15
      WRITE (6,909) NS,NF
  909 FORMAT(1H1,'THERE ARE NO PATHS FROM NODE',I4,' TO NODE',I4)
      GO TO 200
   15 WRITE(6,901) NS,NF
  901 FORMAT(1H1,'THE K SHORTEST PATHS FROM NODE',I4,' TO NODE',I4//1HO,
     1'PATH   LENGTH   NODE SEQUENCE'//)
C
C  THE JJ-TH DISTINCT PATH LENGTH IS BEING EXPLORED.
C
   20 KK=1
      LAB=X(NF,JJ)
      IF(LAB.EQ.INF) GO TO 200
      LL=LAB
      P(1)=NF
   30 LAST=0
C
C  NODES INCIDENT TO NODE P(KK) ARE SCANNED.
C
   40 NT=P(KK)
      IS=START(NT)
      DO 45 ND=NT,1000
      IF(START(ND+1).NE.0) GO TO 48
   45 CONTINUE
   48 IF=START(ND+1)-1
      II=IS+LAST
   50 IF(II.GT.IF) GO TO 90
      NI=INC(II)
      NV=VAL(II)
      LT=LAB-NV
C
C  A TEST IS MADE TO SEE IF THE CURRENT PATH CAN BE EXTENDED BACK TO
C  NODE NI.
C
      DO 60 J=1,K
      IF(X(NI,J)-LT) 60,80,70
   60 CONTINUE
   70 II=II+1
      GO TO 50
   80 KK=KK+1
      IF(KK.GT.5000) GO TO 190
      P(KK)=NI
      Q(KK)=II-IS+1
      PV(KK)=NV
      LAB=LT
C
C  TESTS ARE MADE TO SEE IF THE CURRENT PATH CAN BE EXTENDED FURTHER.
C
      IF(LAB.NE.0) GO TO 30
      IF(NI.NE.NS) GO TO 30
C
C  A COMPLETE PATH FROM NS TO NF HAS BEEN GENERATED AND IS PRINTED
```

```
C  OUT ON UNIT 6.
C
      NP=NP+1
      WRITE (6,902) NP,LL,(P(J),J=KK,1,-1)
  902 FORMAT(1X, I4,I8,5X,(20I5))
      IF(NP.GE.PMAX) GO TO 200
   90 LAST=Q(KK)
      P(KK)=0
      LAB=LAB+PV(KK)
      KK=KK-1
      IF(KK.GT.0) GO TO 40
C
C  THE EXPLORATION OF THE CURRENT JJ-TH DISTINCT PATH LENGTH IS ENDED.
C
      JJ=JJ+1
      IF(JJ.GT.K) GO TO 200
      GO TO 20
  190 WRITE(6,903)
  903 FORMAT(140,'NUMBER OF ARCS IN PATH EXCEEDS 5000')
  200 RETURN
      END
```

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How Bad Is the Hadamard Determinantal Bound?*

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The Hadamard bound for the determinant of an n by n matrix is a good one in that equality may be attained in a rich class of cases. However, the bound generally gives up a good deal, and we answer the title question "on the average." Assuming the entries of $A=(a_{ij})$ are uniformly distributed over some interval symmetric about the origin, the expected value of the ratio of $(\det A)^2$ to the square of the Hadamard bound is found to be $\frac{n!}{n^n}$. The expectations of the square of the Hadamard bound and of $(\det A)^2$ are also computed individually, and their ratio turns out also to be $\frac{n!}{n^n}$.

Key words: Determinant; expected value; Hadamard determinantal bound; uniform distribution.

The Hadamard bound is often used as an upper estimate for the determinant of an n by n matrix in computational algorithms as well as other numerical estimates. It states that for an n by n matrix $A=(a_{ij})$,

$$|\det A| \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \equiv H(A).$$

Since equality may be attained (either when the rows of A are orthogonal or when one row is zero), the bound is both simple and theoretically sound. However, computational experience suggests that the bound generally exaggerates the determinant rather heavily. This is unfortunate when the computation time of an algorithm is proportional to a determinantal estimate. One might well ask what would be a good estimate "on the average."

In order to provide a partial answer, we assume that $A=(a_{ij})$ is a real n by n matrix whose entries are chosen independently from a uniform distribution on $[-1, 1]$. Then $d \equiv \det A$, $H \equiv H(A)$ and d/H may be considered as random variables. If $f=f(A)$ is any scalar-valued function of the entries of the matrix A, we denote the expected value (if it exists) of the random variable f by

$$E(f) \equiv 2^{-n^2} \int_{-1}^{1} f da_{ij}$$

where the notation means that each of the n^2 variables of integration a_{ij} runs over the interval $[-1, 1]$. We shall prove:

THEOREM 1: *We have*

$$E(d^2) = n!/3^n, \tag{i}$$

$$E(H^2) = n^n/3^n, \tag{ii}$$

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$$E(d^2/H^2) = E(d^2)/E(H^2) = n!/n^n. \tag{iii}$$

PROOF: We have

$$\det\ (A) = \sum_{\sigma} \chi(\sigma) a_{1\sigma(1)} \cdot \cdot \cdot a_{n\sigma(n)},$$

where σ runs over all elements of the symmetric group S_n, and $\chi(\sigma)$ is the alternating character. Thus

$$(\det\ (A))^2 = \sum_{\sigma,\tau} \chi(\sigma)\chi(\tau) a_{1\sigma(1)} a_{1\tau(1)} \cdot \cdot \cdot a_{n\sigma(n)} a_{n\tau(n)}.$$

Now let $f = f(A)$ be any scalar-valued function of the variables a_{ij} which is *even* in each variable. Then

$$E(d^2 f) = 2^{-n^2} \sum_{\sigma,\tau} \chi(\sigma)\chi(\tau) \int_{-1}^{1} f \cdot a_{1\sigma(1)} a_{1\tau(1)} \cdot \cdot \cdot a_{n\sigma(n)} a_{n\tau(n)} da_{ij}.$$

Consider the contribution of the term

$$f \cdot a_{1\sigma(1)} a_{1\tau(1)} \cdot \cdot \cdot a_{n\sigma(n)} a_{n\tau(n)}$$

to the integral. If $\sigma(k) \neq \tau(k)$ for some k such that $1 \leq k \leq n$, then this term is an odd function of $a_{k\sigma(k)}$, and so the integral of this term vanishes, since the range of each variable of integration a_{ij} is $-1 \leq a_{ij} \leq 1$. It follows that

$$E(d^2 f) = 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} f a_{1\sigma(1)}^2 \cdot \cdot \cdot a_{n\sigma(n)}^2 da_{ij}.$$

We first make the choice $f = 1$. Then

$$\begin{aligned} E(d^2) &= 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} a_{1\sigma(1)}^2 \cdot \cdot \cdot a_{n\sigma(n)}^2 da_{ij} \\ &= 2^{-n^2} \sum_{\sigma} \left(\frac{2}{3}\right)^n \cdot 2^{n^2-n} \\ &= n!/3^n. \end{aligned}$$

Hence (i) is proved.

We now choose $f = H^{-2}$. Then

$$E(d^2/H^2) = 2^{-n^2} \sum_{\sigma} \int_{-1}^{1} a_{1\sigma(1)}^2 \cdot \cdot \cdot a_{n\sigma(n)}^2 \Big/ \prod_{i=1}^{n} (a_{i1}^2 + \cdot \cdot \cdot + a_{in}^2) da_{ij}.$$

A moment's reflection shows that the value of the integral is the same for each σ, so that

$$\begin{aligned} E(d^2/H^2) &= 2^{-n^2} n! \int_{-1}^{1} \prod_{i=1}^{n} a_{i1}^2/(a_{i1}^2 + \cdot \cdot \cdot + a_{in}^2) da_{ij} \\ &= 2^{-n^2} n! \left\{ \int_{-1}^{1} x_1^2/(x_1^2 + \cdot \cdot \cdot + x_n^2) dx_1 \cdot \cdot \cdot dx_n \right\}^n \end{aligned}$$

$$= 2^{-n^2} n! \left\{ \frac{1}{n} \int_1^1 (x_1^2 + \ldots + x_n^2)/(x_1^2 + \ldots + x_n^2) dx_1 \ldots dx_n \right\}^n$$

$$= 2^{-n^2} n! n^{-n} 2^{n^2}$$

$$= n!/n^n.$$

Hence (iii) is proved.

The proof of (ii) is similar and we omit it.

REMARK 1: An interpretation may be given to part (iii) of theorem 1 by the observation that the quotient $d(A)^2/H(A)^2$ is just the ratio of det AA^T to the product of the diagonal entries of AA^T. (That this ratio is less than or equal to 1 is another version of Hadamard's inequality.)

REMARK 2: If alternatively the a_{ij} are independently and uniformly distributed over $[-M, M]$, we have:

$$E(d^2) = M^{2n} \frac{n!}{3^n} \tag{i'}$$

and

$$E(H^2) = M^{2n} \frac{n^n}{3^n}. \tag{ii'}$$

The fact that

$$E(d^2/H^2) = E(d^2)/E(H^2)$$

could certainly not have been predicted beforehand, and strikes us as a rather remarkable occurrence. Unfortunately, it seems quite difficult to derive similar formulae for

$$E(|d|), E(H), \qquad E(|d|/H).$$

The previous discussion may be generalized directly to any generalized matrix function. Thus if

$$d(\chi, G) = \sum_{\sigma \in G} \chi(\sigma) a_{1\sigma(1)} \ldots a_{n\sigma(n)},$$

where G is any subgroup of S_n and χ any irreducible character on G, then we have

THEOREM 2: *The expected value of* $|\mathrm{d}(\chi, \mathrm{G})|^2$ *is given by*

$$\mathrm{E}(|\mathrm{d}(\chi, \mathrm{G})|^2) = \mathrm{o}(\mathrm{G})/3^n,$$

where o(G) *is the order of* G.

The proof is just as before, except that the character relationship

$$\sum_{\sigma \in G} |\chi(\sigma)|^2 = o(G)$$

comes into play.

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The problem is that of locating a flow-receiving facility in a region, so as to minimize the weighted sum of distances between sources and facility. It is shown here that if a subregion S both generates "sufficiently much" of the region's total flow, and admits entry via specified "gate" points without "too much" circuity, then (a) S contains at least one "near-optimal" location for the facility, and (b) no strictly optimal location can lie "too far" from S.

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Key words: best integration formulas; diophantine approximation; Gaussian quadrature; integration; Monte Carlo; multiple integration; multiple quadrature; numerical analysis; numerical integration; optimal formulas; quadrature.

This paper is an expository survey of the main methods that have been developed for numerical evaluation of multiple integrals. Among the approaches discussed are: the Monte Carlo method and its generalizations; number-theoretical methods, based essentially on the ideas of diophantine approximation and equidistribution modulo 1; the functional analysis approach, in which the quadrature error is regarded as a linear functional and one attempts to minimize its norm; and the classical approach of designing formulas to be exact for polynomials of high degree while using as few values of the integrand as possible. Most of the research in this field is quite recent.

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Key words: commutator; eigenvalues; positive definite; trace.

It is known that a square matrix A can be written as a commutator $XY - YX$ if and only if $\mathrm{Tr}(A) = 0$. In this note it is shown further that for a fixed A the spectrum of one of the factors may be taken to be arbitrary while the spectrum of the other factor is arbitrary as long as the characteristic roots are distinct. The distinctness restriction on one of the factors may not in general be relaxed.

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Let A be an n-square positive definite hermitian matrix and $A(i|j)$ be the submatrix of A obtained by deleting row i and column j. Let $P(A)$ be the n-square matrix whose i,j entry is per $A(j|i)$. If $D(A)$ is the classical adjoint of A then $n(\mathrm{per}\ A)\ D(A) - (\det A)\ P(A)$ is positive semidefinite.

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An error theory is constructed for the method of stationary phase for integrals of the form

$$I(x) = \int_a^b e^{ixp(t)} q(t)\,dt.$$

Here x is a large real parameter, the function $p(t)$ is real, and neither $p(t)$ nor $q(t)$ need be analytic in t. For both finite and infinite ranges of integration, explicit expressions are derived for the truncation errors associated with the asymptotic expansion of $I(x)$. The use of these explicit expressions for the computation of realistic error bounds is illustrated by means of an example.

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The results of an evaluation of linear least-squares computer programs (Wampler (174, 175)) are briefly summarized. Subsequent work, including problems encountered, to provide a more accurate least-squares routine for the OMNITAB II program is discussed. In this connection, the comparative results of running a number of ill-conditioned problems on OMNITAB II and two other reliable programs are presented. A bibliography of the recent literature on least-squares analysis has been prepared with special emphasis on the computational aspects of obtaining least-squares solutions.

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